INTRODUCTION TO THE OPTIMAL CONTROL OF SYSTEMS WITH DISTRIBUTED PARAMETERS IV. OPTIMALITY CONDITION IN OPTIMAL CONTROL OF PARABOLIC PDES

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1. Introduction and Statement of the Optimal Control Problem

This is the fourth lecture on the optimal control of systems with distributed parameters. Let us first recall the problem. It is required to minimize the functional

$$J(v) = \int_0^l |u(x,T;v) - h(x)|^2 \, dx \tag{1}$$

on a control set

$$V = \Big\{ v = (p,f) \in H = L_2[0,T] \times L_2(Q); p_{\min} \le p \le p_{\max} \text{ a.e. } 0 \le t \le T; \|f\|_{L_2(Q)} \le R \Big\} \tag{2}$$

where $u = u(x,t;v) \in V_2^{1,0}(Q)$ solves the problem

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f, \quad \text{in } Q = \{(x,t) : 0 < x < l, \ 0 < t \le T\} \tag{3}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \quad \frac{\partial u(l,t)}{\partial x} = \nu[p(t) - u(l,t)], \quad 0 < t \le T \tag{4}$$

$$u(x,0) = \phi(x), \ 0 \le x \le l \tag{5}$$

where $p_{\min} < p_{\max}$, $0 < R$, ϕ, w are given. Recall that $u \in V_2^{1,0}(Q)$ is a solution of (3)–(5) if

$$\int_0^T \int_0^l \Big(-u\psi_t + a^2 u_x \psi_x - f\psi \Big) \, dx \, dt + \int_0^l u(x,T)\psi(x,T) \, dx -$$
$$- \int_0^l \phi(x)\psi(x,0) \, dx - a^2 \nu \int_0^T [p(t) - u(l,t)] \, \psi(l,t) \, dt = 0$$

for all $\psi \in W_2^{1,1}(Q)$. On the Hilbert space H of elements $v = \Big(p(t), f(x,t)\Big)$ define the inner product

$$\langle v_1, v_2 \rangle = \int_0^T p_1(t) p_2(t) \, dt + \int_0^T \int_0^l f_1(x,t) f_2(x,t) \, dx \, dt \tag{6}$$

and the norm

$$\|v\|_H = \langle v, v \rangle^{1/2} = \Big(\|p\|_{L_2}^2 + \|f\|_{L_2}^2 \Big)^{1/2}$$

In this lecture we will prove the convexity of the functional J, give a necessary and sufficient optimality condition for a general problem $J(u) \to \inf$ on a convex subset U of an Hilbert space, and discuss the application of this optimality condition to the problem (1)–(5).

Recall the following

Definition 1 (Convexity of a set). A set U is called *convex* if for arbitrary $u, v \in U$ and any $\alpha \in [0,1]$, it follows that

$$\alpha u + (1-\alpha) v \in U$$

Definition 2 (Convexity of a function). Let U be a convex set. A function $J : U \to R$ is called *convex* on U if

$$J\Big(\alpha u + (1-\alpha)v\Big) \leq \alpha J(u) + (1-\alpha)J(v) \tag{7}$$

for all $u, v \in U$ and all $\alpha \in [0,1]$. J is called ***strictly convex*** if equality in (7) is only possible when $\alpha = 0$ or $\alpha = 1$.

2. Convexity of the cost functional J

First of all, observe that the solution $u = u(x,t;v)$ of the PDE problem (3)–(5) depends linearly on v, since the problem is linear and the solution in $V_2^{1,0}(Q)$ is unique. In particular,

$$u(x,t;\alpha v + (1-\alpha)w) = \alpha u(x,t;v) + (1-\alpha)u(x,t;w)$$

Then we have

$$\begin{aligned}
J(\alpha v + (1-\alpha)w) &= \int_0^l |u(x,T;\alpha v + (1-\alpha)w) - h(x)|^2 \, dx \\
&= \int_0^l |\alpha u(x,t;v) + (1-\alpha)u(x,t;w) - h(x)|^2 \, dx \\
&= \int_0^l \Big|\alpha\Big(u(x,t;v) - h(x)\Big) + (1-\alpha)\Big(u(x,t;w) - h(x)\Big)\Big|^2 \, dx \\
&= \alpha^2 \langle u(x,T;v) - h(x), u(x,T;v) - h(x)\rangle + \\
&\quad + (1-\alpha)^2 \langle u(x,T;w) - h(x), u(x,T;w) - h(x)\rangle + \\
&\quad + 2\alpha(1-\alpha) \langle u(x,T;v) - h(x), u(x,T;w) - h(x)\rangle \\
&= \alpha J(v) + (1-\alpha)J(w) - \alpha(1-\alpha) \langle u(x,T;v) - h(x), u(x,T;v) - h(x)\rangle - \\
&\quad - \alpha(1-\alpha) \langle u(x,T;w) - h(x), u(x,T;w) - h(x)\rangle + \\
&\quad + 2\alpha(1-\alpha) \langle u(x,T;v) - h(x), u(x,T;w) - h(x)\rangle
\end{aligned}$$

so

$$\begin{aligned}
J(\alpha v + (1-\alpha)w) = \alpha J(v) + (1-\alpha)J(w) - \\
- \alpha(1-\alpha) \langle u(x,T;v) - u(x,T;w), u(x,T;v) - u(x,T;w)\rangle
\end{aligned}$$

That is,

$$J(\alpha v + (1-\alpha)w) = \alpha J(v) + (1-\alpha)J(w) - \alpha(1-\alpha) \|u(x,T;v-w)\|^2_{L_2[0,l]} \tag{8}$$

In particular,

$$J(\alpha v + (1-\alpha)w) \leq \alpha J(v) + (1-\alpha)J(w) \ \forall v, w \in V, \ \alpha \in [0,1]$$

We have proved that the functional $J(v)$ is conves on V.

3. Optimality Conditions in Banach Space

We will now derive necessary and sufficient conditions for minimality of functionals over Banach spaces; they are of the same spirit as the familiar optimality conditions in finite dimensional space, but established in a general context.

Theorem 1. *Let U be a convex subset of the Banach space B, and consider the problem*

$$J(u) \to \min \text{ on } U \tag{9}$$

Suppose $J(u) \in C^1(U)$, and denote by

$$U_* = \Big\{ u \in U : J(u) = J_* = \inf_U J(u) \Big\} \tag{10}$$

the set of solutions to (9). *For every $u_* \in U_*$, the following necessary condition is satisfied:*

$$\langle J'(u_*), u - u_* \rangle \geq 0 \; \forall u \in U \tag{11}$$

If $u_* \in int(U)$, condition (11) *is equivalent to*

$$J'(u_*) = 0 \tag{12}$$

Moreover, if $J(u)$ is convex on U, then (11) *is also a sufficient condition to assert that the element $u_* \in U_*$.*

Proof.

Necessity: Let $u_* \in U_*$, then for every $u \in U$, and every $\alpha \in [0,1]$ we have

$$0 \leq J\Big(u_* + \alpha(u - u_*)\Big) - J(u_*) = \langle J'(u_*), \alpha(u - u_*) \rangle + o(\alpha)$$

Dividng through by α we have

$$0 \leq \langle J'(u_*), u - u_* \rangle + \frac{o(\alpha)}{\alpha}$$

Passing to the limit as $\alpha \downarrow 0$, (11) follows.

If $u_* \in \text{int}U_*$, then for all $e \in B$ with $\|e\| = 1$ there exists $\epsilon_0 > 0$ such that $u_* + \epsilon e \in U$ for $|\epsilon| < \epsilon_0$. Hence

$$0 \leq \langle J'(u_*), \epsilon e \rangle = \epsilon \, \langle J'(u_*), e \rangle$$

By taking $0 < \epsilon < \epsilon_0$ or $-\epsilon_0 < \epsilon < 0$ and dividing through by ϵ it follows that

$$0 = \langle J'(u_*), e \rangle$$

Since e is arbitrary, $J'(u_*)$ must be the zero element in B^*.

Sufficiency: Assume that $J(u) \in C^1(U)$ is convex on U and for some $u_* \in U$, (11) is satisfied. We will prove sufficiency with the help of the following

Claim 1. *$J(u) \in C^1(U)$ is convex if and only if*

$$J(u) \geq J(v) + \langle J'(v), u - v \rangle \;\; \forall u, v \in U \tag{13}$$

Proof.

"only if": If $J(u) \in C^1(U)$ is convex,

$$J\Big(\alpha u + (1-\alpha)v\Big) \leq \alpha J(u) + (1-\alpha)J(v)$$

Therefore

$$J\Big(\alpha u + (1-\alpha)v\Big) - J(v) \leq \alpha\Big(J(u) - J(v)\Big)$$

By definition of Frechet differentiability, the left-hand side can be rewritten as

$$\Big\langle J'\Big(v+\theta\alpha(u-v)\Big),\alpha(u-v)\Big\rangle \le \alpha\Big(J(u)-J(v)\Big)$$

Divide both sides by α and pass to the limit as $\alpha \downarrow 0$ to derive

$$\langle J'(v), u-v\rangle \le J(u)-J(v)$$

which implies (13)

"if": Fix $u, v \in U$ and define

$$u_\alpha = \alpha u + (1-\alpha)v$$

Suppose $J(u) \in C^1(U)$ satisfies (13). Then

$$J(u)-J(u_\alpha) \ge \langle J'(u_\alpha), u-u_\alpha\rangle \; J(v)-J(u_\alpha) \ge \langle J'(u_\alpha), v-u_\alpha\rangle$$

Multiplying the former inequality by α, the latter by $1-\alpha$, and adding gives

$$\alpha J(u)+(1-\alpha)J(v)-J(u_\alpha) \ge \langle J'(u_\alpha), u_\alpha-u_\alpha\rangle = 0$$

It follows that

$$J(u_\alpha) \le \alpha J(u)+(1-\alpha)J(v)$$

which is (7)

Claim is proved. □

Now, by choosing $v = u_*$ in (13) we have

$$J(u) \ge J(u_*) + \langle J'(u_*), u-u_*\rangle\,,\ \forall u \in U$$

and by (11) we have

$$J(u) \ge J(u_*),\ \forall u \in U$$

That is, $u_* \in U_*$. □

4. Optimality Condition applied to (1)–(5)

Recall that we proved the following

Theorem 2. *When $\phi \in L_2[0,l]$, $J(v)$ is Frechet differentiable at $v \in V$ and*

$$J'(v) = \Big(a^2\nu\psi(l,t), \psi(x,t)\Big) \in H \tag{14}$$

where $\psi = \psi(x,t) \in V_2^{1,0}(Q)$ is a weak solution of the problem

$$\psi_t + a^2\psi_{xx} = 0, \quad 0<x<l,\ 0\le t<T \tag{15}$$

$$\psi_x(0,t) = 0, \quad \psi_x(l,t)+\nu\psi(l,t) = 0, \quad 0<t<T \tag{16}$$

$$\psi(x,T) = 2\Big(u(x,T;v)-h(x)\Big), \quad 0\le x\le l \tag{17}$$

Since $J(v)$ is convex, it follows that the necessary and sufficient condition for the optimality of a control

$$v_* = \Big(p_*(t), f_*(x,t)\Big) \in U$$

is the following inequality

$$0 \le \langle J'(u_*), u - u_* \rangle_H = \int_0^T a^2 \nu \psi(l,t;u_*)\Big(p(t) - p_*(t)\Big)\,dt + \int_0^T \int_0^l \psi(x,t;u_*)\Big(f(x,t) - f_*(x,t)\Big)\,dx\,dt$$

for all $v = \Big(p(t), f(x,t)\Big) \in U$. Note that $\psi \in V_2^{1,0}(Q)$ is a weak solution of the adjoint problem (15)–(17); from the proof of Frechet differentiability of J it follows that an equivalent formulation of the necessary and sufficient condition is

$$\int_0^l \Big(u(x,T;u_*) - h(x)\Big)\Big(u(x,T;v) - u(x,T;u_*)\Big)\,dx \ge 0 \ \forall v \in V$$

References

[1] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uraltseva, ***Linear and Quasilinear Equations of Parabolic Type***, vol. 23 of Translations of Mathematical Monographs, American Mathematical Society, Providence, R. I., 1968.

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