INTRODUCTION TO OPTIMAL CONTROL FOR SYSTEMS WITH DISTRIBUTED PARAMETERS. II. FRECHET DIFFERENTIABILITY IN OPTIMAL CONTROL OF PARABOLIC PDES - PART 3

UGUR ABDULLA

1. Introduction and Statement of the Optimal Control Problem

This is the third lecture on the optimal control of systems with distributed parameters. The aims of this lecture are twofolds. First I am going to demonstrate that the general method we used to prove the Frechet differentiability and to derive the formula for the Frechet gradiaent is applicable to optimal control problem with constraints. Second, by looking to this new constrained optimal control problem I am going to present general heuristic argument for the derivation of the Frechet gradient which is applicable to optimal control problems for PDEs in a very broad context. In fact, we will see that the solution ψ of the adjoint problem is just infinite-dimensional analogy of Lagrange multipliers arising in constrained optimization problems.

Let us first recall the problem. It is required to minimize the functional

$$J(v) = \int_0^l |u(x,T;v) - w(x)|^2 \, dx \tag{1}$$

on a control set

$$V = \Big\{ v = (p,f) : p \in L_2[0,T],\ f \in L_2(Q); p_{\min} \le p \le p_{\max} \text{ a.e. } 0 \le t \le T; \|f\|_{L_2(Q)} \le R \Big\} \tag{2}$$

where $u = u(x,t;v) \in V_2^{1,0}(Q)$ solves the problem

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f, \quad \text{in } Q = \{(x,t) : 0 < x < l,\ 0 < t \le T\} \tag{3}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \quad \frac{\partial u(l,t)}{\partial x} = \nu[p(t) - u(l,t)], \quad 0 < t \le T \tag{4}$$

$$u(x,0) = \phi(x),\ 0 \le x \le l \tag{5}$$

where $p_{\min} < p_{\max}$, $0 < R$, ϕ, w are given. That is,

$$\int_0^T \int_0^l \Big(-u\psi_t + a^2 u_x \psi_x - f\psi \Big) \, dx \, dt + \int_0^l u(x,T)\psi(x,T) \, dx -$$
$$- \int_0^l \phi(x)\psi(x,0) \, dx - a^2 \nu \int_0^T [p(t) - u(l,t)] \, \psi(l,t) \, dt = 0$$

for all $\psi \in W_2^{1,1}(Q)$. Denote by $H = L_2[0,T] \times L_2(Q)$ the Hilbert space of elements $v = \Big(p(t), f(x,t)\Big)$ with inner product

$$\langle v_1, v_2 \rangle = \int_0^T p_1(t) p_2(t) \, dt + \int_0^T \int_0^l f_1(x,t) f_2(x,t) \, dx \, dt \tag{6}$$

Date: April 27, 2015.

with the norm

$$\|v\|_H = \langle v, v\rangle^{1/2} = \Big(\|p\|_{L_2}^2 + \|f\|_{L_2}^2\Big)^{1/2}$$

Last time, we proved the following

Theorem 1. *When $\phi \in L_2[0,l]$, $J(v)$ is Frechet differentiable at $v \in V$ and*

$$J'(v) = \Big(a^2\nu\psi(l,t), \psi(x,t)\Big) \in H \tag{7}$$

where $\psi = \psi(x,t) \in V_2^{1,0}(Q)$ is a weak solution of the problem

$$\psi_t + a^2\psi_{xx} = 0, \quad 0 < x < l, \; 0 \le t < T \tag{8}$$

$$\psi_x(0,t) = 0, \quad \psi_x(l,t) + \nu\psi(l,t) = 0, \quad 0 < t < T \tag{9}$$

$$\psi(x,T) = 2\Big(u(x,T;v) - w(x)\Big), \quad 0 \le x \le l \tag{10}$$

2. Constrained Optimal Control Problem

Consider a more difficult problem: it is required that the temperature within the media is bounded by some constant value:

$$u(x,t;v) \le \bar{u} \tag{11}$$

Physical meaning of this constrained is that temperature of the media should be kept below particular threshhold value. This is very important in many applications. In particular, note that one of the control parameters is f - density of the heat sources, which can be used to increase the temperature and one wants to avoid overheating.

Let us apply the Penalty Method to solve the constrained optimal control problem (1)–(5), (11). Consider a penalty functional as follows

$$P_k(v) = A_k \int_0^T \int_0^l |\max\{u(x,t;v) - \bar{u}; 0\}|^2 \, dx\,dt \tag{12}$$

where $\{A_k\}$ is a given positive sequence with $A_k \to \infty$ as $k \to \infty$. Consider a minimization problem

$$\Phi_k(v) = J(v) + P_k(v) \to \min \tag{13}$$

on V under the condition (3)–(5). The idea of the method is that the control which violates the constraint (11) is penalized through increase of the cost functional.

By applying the method used in the non-constrained case we can prove that the functional (13) is Frechet differentiable on H and its gradient is

$$\Phi_k'(v) = \Big(a^2\nu\psi_k(l,t;v), \psi_k(x,t;v)\Big) \in H \tag{14}$$

where $\psi_k(x,t;v)$ is a weak solution of the adjoint PDE

$$\psi_t + a^2\psi_{xx} = -2A_k \max\{u(x,t;v) - \bar{u}; 0\}, \quad 0 < x < l, \; 0 \le t < T \tag{15}$$

$$\psi_x(0,t) = 0, \quad \psi_x(l,t) + \nu\psi(l,t) = 0, \quad 0 < t < T \tag{16}$$

$$\psi(x,T) = 2\Big(u(x,T;v) - w(x)\Big), \quad 0 \le x \le l \tag{17}$$

Indeed, for the increment of the functional

$$\Phi_k(v) = \Phi_k(v + \Delta v) - \Phi_k(v) = \Delta J(v) + \Delta P_k(v)$$

we have

$$\Delta J_k(v) = \int_0^l 2\Big(u(x,T;v) = w(x)\Big)\Delta u(x,T)\,dx + \int_0^T |\Delta u(x,T)|^2\;dx$$

$$\Delta P_k(v) = P_k(v+\Delta v) - P_k(v) = A_k \int_0^T \int_0^l |\max\{u(x,t;v+\Delta v) - \bar{u};0\}|^2\;dx\,dt -$$

$$-A_k \int_0^T \int_0^l |\max\{u(x,t;v) - \bar{u};0\}|^2\;dx\,dt = A_k \int_0^T \int_0^l |\max\{u(x,t;v) - \bar{u} + \Delta u(x,t);0\}|^2\;dx\,dt -$$

$$-A_k \int_0^T \int_0^l |\max\{u(x,t;v) - \bar{u};0\}|^2\;dx\,dt$$

Since the function

$$g(z) = \max\{z;0\}^2 = \begin{cases} 0, & z < 0 \\ z^2, & z \geq 0 \end{cases}$$

is continuously differentiable with

$$g'(z) = \begin{cases} 0, & z < 0 \\ 2z, & z \geq 0 \end{cases}$$

we have

$$\Delta P_k(v) = 2A_k \int_0^T \int_0^l \max\{u(x,t;v) - \bar{u};0\}\,\Delta u(x,t)\,dx\,dt + R_k$$

$$R_k = 2A_k \int_0^T \int_0^l \Big[\max\{u(x,t;v) - \bar{u} + \theta\Delta u(x,t);0\} - \max\Big(u(x,t;v) - \bar{u};0)\Big)\Big]\,\Delta u(x,t)\,dx\,dt$$

where $0 < \theta < 1$ is in general a function of (x,t). Note that the derivative $g(z)$ is in fact Lipschitz:

$$|g'(z+\Delta z) - g'(z)| \leq 2\,|\Delta z|$$

so we have the estimate

$$|R_k| \leq 2A_k \int_0^T \int_0^l |\Delta u(x,t)|^2\;dx\,dt$$

Hence we have

$$\Phi_k(v) = \int_0^l 2\Big(u(x,T;v) - w(x)\Big)\Delta u(x,T)\,dx + \int_0^T \int_0^l 2A_k \max\{u(x,t;v) - \bar{u};0\}\,\Delta u(x,t)\,dx\,dt + \tilde{R}_k$$

Where

$$\left|\tilde{R}_k\right| \leq \int_0^T |\Delta u(x,T)|^2\;dx + 2A_k \int_0^T \int_0^l |\Delta u(x,t)|^2\;dx\,dt$$

Now we can derive the formula for the Frechet gradient of the functional $\Phi_k(v)$ given above with the same procedure as for $J(v)$, but let us instead focus on another question: Where is the adjoint problem coming from?

3. General Heuristic Argument, Connection of the Adjoint Problem with Lagrange Multipliers

Let $v=(p,f)\in H=L_2[0,T]\times L_2(Q)$ be given. We build the Lagrange functional:

$$\mathcal{L}(u,p,f,\psi)=\int_0^l |u(x,t;v)-w(x)|^2\,dx+A_k\int_0^T\int_0^l \Big|\max\Big(u(x,t;v)-\bar{u};0\Big)\Big|^2+ \\ +\int_0^T\int_0^l \psi(x,t)\Big(-u_t(x,t)+a^2u_{xx}+f(x,t)\Big)\,dx\,dt \tag{18}$$

where $\psi(x,t)$ is the Lagrange multiplier corresponding to the PDE constraint. Assume that $u(x,t)$ and $\psi(x,t)$ are sufficiently smooth on $\bar{Q}$; since the PDE is already taken into account in (18), we will only assume that $u(x,t)$ and $p(t)$ satisfy initial-boundary conditions (4), (5). Aditional conditions on ψ will be given below.

Let us give an increment (or variation) to the variables u, p, and f, that is, consider

$$u(x,t)+\delta u(x,t),\ p(t)+\delta p(t),\ f(x,t)+\delta f(x,t)$$

which satisfy initial boundary conditions (4), (5), so

$$\delta u_x(0,t)=0,\quad \delta u_x(l,t)=\nu\Big(\delta p(t)-\delta u(l,t)\Big),\quad \delta u(x,0)=0$$

Calculate the first variation (or main linear part of the increment) of the Lagrange functional:

$$\delta\mathcal{L}=\int_0^l 2\Big(u(x,T;v)-w(x)\Big)\delta u(x,T)\,dx+\int_0^T\int_0^l 2A_k\max\Big(u(x,t;v)-\bar{u};0\Big)\delta u(x,t)\,dx\,dt+ \\ +\int_0^T\int_0^l \psi(x,t)\Big(-\delta u_t(x,t)+a^2\delta u_{xx}(x,t)+\delta f(x,t)\Big)\,dx\,dt$$

Integration by parts (or Gauss-Green formula) gives

$$\delta\mathcal{L}=\int_0^l 2\Big(u(x,T;v)-w(x)\Big)\delta u(x,T)\,dx+\int_0^T\int_0^l 2A_k\max\Big(u(x,t;v)-\bar{u};0\Big)\delta u(x,t)\,dx\,dt+ \\ +\int_0^T\int_0^l \Big(\psi_t(x,t)+a^2\psi_{xx}(x,t)\Big)\,\delta u(x,t)\,dx\,dt+\int_0^T\int_0^l \psi(x,t)\delta f(x,t)\,dx\,dt- \\ -\int_0^l \psi(x,T)\delta u(x,T)\,dx+\int_0^l \psi(x,0)\delta u(x,0)+a^2\int_0^T \psi(l,t)\delta u_x(l,t)\,dt- \\ -a^2\int_0^T \psi(0,t)\delta u_x(0,t)\,dt-a^2\int_0^T \psi_x(l,t)\delta u(l,t)\,dt+a^2\int_0^T \psi_x(0,t)\delta u(0,t)\,dt$$

Applying the conditions on the variation δu, this implies

$$\delta\mathcal{L}=\int_0^l 2\Big(u(x,T;v)-w(x)\Big)\delta u(x,T)\,dx+\int_0^T\int_0^l 2A_k\max\Big(u(x,t;v)-\bar{u};0\Big)\delta u(x,t)\,dx\,dt+ \\ +\int_0^T\int_0^l \Big(\psi_t(x,t)+a^2\psi_{xx}(x,t)\Big)\,\delta u(x,t)\,dx\,dt+\int_0^T\int_0^l \psi(x,t)\delta f(x,t)\,dx\,dt- \\ -\int_0^l \psi(x,T)\delta u(x,T)\,dx+a^2\nu\int_0^T \psi(l,t)\Big(\delta p(t)-\delta u(l,t)\Big)\,dt- \\ -a^2\int_0^T \psi_x(l,t)\delta u(l,t)\,dt+a^2\int_0^T \psi_x(0,t)\delta u(0,t)\,dt$$

Collecting like terms,

$$\delta\mathcal{L} = \int_0^l \left[2\Big(u(x,T;v) - w(x)\Big) - \psi(x,T)\right] \delta u(x,T)\, dx +$$
$$+ \int_0^T \int_0^l \Big(\psi_t(x,t) + a^2\psi_{xx}(x,t) + 2A_k \max\Big(u(x,t;v) - \bar{u};0\Big)\Big)\, \delta u(x,t)\, dx\, dt +$$
$$\int_0^T \int_0^l \psi(x,t)\delta f(x,t)\, dx\, dt + a^2\nu \int_0^T \psi(l,t)\delta p(t)\, dt -$$
$$-a^2 \int_0^T (\psi_x(l,t) + \nu\psi(l,t))\, \delta u(l,t)\, dt + a^2 \int_0^T \psi_x(0,t)\delta u(0,t)\, dt$$

Assuming that at the optimal point $\delta\mathcal{L} = 0$, and due to the arbitraricity of $\delta u(x,t)$, the coefficients of $\delta u(x,T)$, $\delta u(x,t)$, $\delta u(l,t)$, and $\delta u(0,t)$ must be zero. This gives he adjoint problem; making the coefficients on δp, δf equal to zero we derive the optimality condition

$$\psi(l,t) = 0, \quad \psi(x,t) = 0$$

that is, $\Phi_k'(v) = 0$, for $v \in V = H$, in view of (14).

References

[1] F.P. Vasilev, *Optimization Methods: Minimization in Functional Spaces, Regularization and Approximation*, Moscow, 2002

[2] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, vol. 23 of Translations of Mathematical Monographs, American Mathematical Society, Providence, R. I., 1968.

E-mail address: abdulla@fit.edu

Mathematical Sciences Department, Florida Institute of Technology, 150 W. University Blvd., Melbourne, FL 32901