INTRODUCTION TO OPTIMAL CONTROL FOR SYSTEMS WITH DISTRIBUTED PARAMETERS. II. FRECHET DIFFERENTIABILITY IN OPTIMAL CONTROL OF PARABOLIC PDES - PART 2

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1. Introduction and Statement of the Optimal Control Problem

This is the second lecture on the optimal control of systems with distributed parameters. I am going to continue proof of the formula for the Frechet gradient of the cost functional in the problem of optimal control of parabolic PDEs. Let me remind the problem. It is required to minimize the functional

$$J(v) = \int_0^l |u(x,T;v) - w(x)|^2 \, dx \tag{1}$$

on a control set

$$V = \Big\{ v = (p,f) : p \in L_2[0,T],\ f \in L_2(Q); p_{\min} \le p \le p_{\max} \text{ a.e. } 0 \le t \le T; \|f\|_{L_2(Q)} \le R \Big\} \tag{2}$$

where $u = u(x,t;v)$ solves the problem

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f, \quad \text{in } Q = \{(x,t) : 0 < x < l,\ 0 < t \le T\} \tag{3}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \quad \frac{\partial u(l,t)}{\partial x} = \nu[p(t) - u(l,t)], \quad 0 < t \le T \tag{4}$$

$$u(x,0) = \phi(x),\ 0 \le x \le l \tag{5}$$

where $p_{\min} < p_{\max}$, $0 < R$, ϕ, w are given.

Denote by $H = L_2[0,T] \times L_2(Q)$ the Hilbert space of elements $v = \Big(p(t), f(x,t)\Big)$ with inner product

$$\langle v_1, v_2 \rangle = \int_0^T p_1(t) p_2(t) \, dt + \int_0^T \int_0^l f_1(x,t) f_2(x,t) \, dx \, dt \tag{6}$$

with the norm

$$\|v\|_H = \langle v, v \rangle^{1/2} = \Big(\|p\|_{L_2}^2 + \|f\|_{L_2}^2 \Big)^{1/2}$$

Let $\phi \in L_2[0,l]$. Given $v \in V$, the function $u = u(x,t;v) \in V_2^{1,0}(Q)$ is called a weak solution of (3)–(5) if

$$\int_0^T \int_0^l \Big(-u\psi_t + a^2 u_x \psi_x - f\psi \Big) \, dx \, dt + \int_0^l u(x,T)\psi(x,T) \, dx -$$
$$- \int_0^l \phi(x)\psi(x,0) \, dx - a^2 \nu \int_0^T \Big[p(t) - u(l,t) \Big] \psi(l,t) \, dt = 0 \tag{7}$$

for all $\psi \in W_2^{1,1}(Q)$.

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Let us continue the proof of differentiability of $J(v)$ in our problem. Choose any $v=(p,f)$ and let $\Delta v=(\Delta p,\Delta f)$ so $v+\Delta v=(p+\Delta p,f+\Delta f)$, and let

$$\Delta u(x,t)=u(x,t;v+\Delta v)-u(x,t;v)$$

Then Δu is a solution from $V_2^{1,0}(Q)$ of the problem

$$\frac{\partial \Delta u}{\partial t}=a^2\frac{\partial^2\Delta u}{\partial x^2}+\Delta f \quad \text{in } Q \tag{8}$$

$$\frac{\partial \Delta u(0,t)}{\partial x}=0, \quad \frac{\partial \Delta u(l,t)}{\partial x}=\nu[\Delta p(t)-\Delta u(l,t)],\ 0<t\le T \tag{9}$$

$$\Delta u(x,0)=0, \quad 0\le x\le l \tag{10}$$

It satisfies the integral identity

$$\int_0^T\int_0^l\Big(-\Delta u\psi_t+a^2\Delta u_x\psi_x-\Delta f\psi\Big)\,dx\,dt+\int_0^l\Delta u(x,T)\psi(x,T)\,dx-$$
$$-a^2\nu\int_0^T\Big[\Delta p(t)-\Delta u(l,t)\Big]\psi(l,t)\,dt=0 \quad \forall\psi\in W_2^{1,1}(Q) \tag{11}$$

We can transform ΔJ as follows:

$$\Delta J=J(v+\Delta v)-J(v)=\int_0^l|u(x,t;v+\Delta v)-w(x)|^2\,dx-\int_0^l|u(x,T;v)-w(x)|^2\,dx=$$
$$=\int_0^l 2\left(u(x,T;v)-w(x)\right)\Delta u(x,T)\,dx+\int_0^l|\Delta u(x,T)|^2\,dx$$

Assume that the function $\psi(x,t)$ is chosen such that

$$\psi(x,T)=2\Big(u(x,T;v)-w(x)\Big) \tag{12}$$

so

$$\int_0^l 2\left(u(x,T;v)-f(x)\right)\Delta u(x,T)\,dx=\int_0^l\psi(x,T)\Delta u(x,T)\,dx$$

From the integral identity (11) it follows that

$$\int_0^l\psi(x,T)\Delta u(x,T)\,dx=\int_0^T\int_0^l\Big(\Delta u\psi_t-a^2\Delta u_x\psi_x+\Delta f\psi\Big)\,dx\,dt+$$
$$+a^2\nu\int_0^T[\Delta p(t)-\Delta u(l,t)]\psi(l,t)\,dt=a^2\nu\int_0^T\psi(l,t)\Delta p(t)\,dt+\int_0^T\int_0^l\psi(x,t)\Delta f(x,t)\,dx\,dt+$$
$$+\int_0^T\int_0^l\Big(\Delta u\psi_t-a^2\Delta u_x\psi_x\Big)\,dx\,dt-a^2\nu\int_0^T\psi(l,t)\Delta u(l,t)\,dt \tag{13}$$

The first two integrals on the RHS look like the main linear part of the increment ΔJ, and would provide the formula for the gradient as

$$J'(v)=\Big(a^2\nu\psi(l,t),\psi(x,t)\Big)\in H \tag{14}$$

if we can choose the function ψ to satisfy (12) and

$$\int_0^T\int_0^l\Big(\Delta u\psi_t-a^2\Delta u_x\psi_x\Big)\,dx\,dt-a^2\nu\int_0^T\psi(l,t)\Delta u(l,t)\,dt \tag{15}$$

Hence, in order to prove the formula for the gradient $J'(v)$ as in (14), we have to find a function ψ which satisfies condition (12) and the integral identity (15). Does such ψ exist? It looks like ψ

can be chosen as a weak solution of some parabolic PDE problem. To see that, let us transform (15) assuming that ψ has enough regularity:

$$\int_0^T \int_0^l \left(\Delta u \psi_t - a^2 \Delta u_x \psi_x\right) dx\, dt - a^2 \nu \int_0^T \psi(l,t) \Delta u(l,t)\, dt =$$
$$= \int_0^T \int_0^l \Delta u \left(\psi_t + a^2 \psi_{xx}\right) dx\, dt - a^2 \int_0^T \Delta u(l,t) \psi_x(l,t)\, dt +$$
$$+ a^2 \int_0^T \Delta u(0,t) \psi_x(0,t)\, dt - a^2 \nu \int_0^T \psi(l,t) \Delta u(l,t)\, dt =$$
$$0 = \int_0^T \int_0^l \Delta u \left(\psi_t + a^2 \psi_{xx}\right) dx\, dt - a^2 \int_0^T \left(\psi_x(l,t) + \nu \psi(l,t)\right) dt +$$
$$+ a^2 \int_0^T \psi_x(0,t) \Delta u(0,t)\, dt$$

This clearly indicates that if we choose ψ as a solution of the adjoint problem

$$\psi_t + a^2 \psi_{xx} = 0, \quad 0 < x < l,\ 0 \le t < T \tag{16}$$
$$\psi_x(0,t) = 0, \quad \psi_x(l,t) + \nu \psi(l,t) = 0, \quad 0 < t < T \tag{17}$$
$$\psi(x,T) = 2\Big(u(x,T;v) - w(x)\Big), \quad 0 \le x \le l \tag{18}$$

then we express the gradient through ψ via the formula (14). Hence, what we need is to choose a solution of (16)–(18) which satisfies the integral identity (15). That means we need a solution in $W_2^{1,1}(Q)$:

Definition 1. $\psi \in W_2^{1,1}(Q)$ is called a weak solution of the problem (16)–(18) if ψ satisfies (18) and

$$\int_0^T \int_0^l \left(\eta \psi_t - a^2 \eta_x \psi_x\right) dx\, dt - a^2 \nu \int_0^T \psi(l,t) \eta(l,t)\, dt = 0$$

for every $\eta \in W_2^{1,0}(Q)$.

However, this solution may not exist, since

$$2\Big(u(x,T;v) - w(x)\Big) \in L_2[0,l]$$

but the Sobolev trace embedding theorem implies that

$$W_2^{1,1}(Q) \hookrightarrow W_2^{1/2}[0,l],$$

that is to say, for any function $\psi \in W_2^{1,1}(Q)$, its trace $\psi(x,T)$ belongs to $W_2^{1/2}[0,l]$. Since boundary function in (18) is only in $L_2[0,l]$, in general there is no solution $\psi \in W_2^{1,1}(Q)$ of the adjoint problem (16)–(18). However, there exists a unique solution of (16)–(18) in $V_2^{1,0}(Q)$:

Definition 2. $\psi \in V_2^{1,0}(Q)$ is a weak solution of (16)–(18) if

$$\int_0^T \int_0^l \left(-\eta_t \psi - a^2 \eta_x \psi_x\right) dx\, dt - a^2 \nu \int_0^T \psi(l,t) \eta(l,t)\, dt +$$
$$+ \int_0^l 2\Big(u(x,T;v) - f(x)\Big) \eta(x,T)\, dx - \int_0^l \psi(x,0) \eta(x,0)\, dx$$

for all $\eta \in W_2^{1,1}(Q)$.

From [1] we know that there exists a unique solution ψ of (16)–(18) in $V_2^{1,0}(Q)$ which satisfies the energy estimate

$$\|\psi\|_{V_2^{1,0}(Q)} \leq C \left\|2\left(u(x,T;v)-w(x)\right)\right\|_{L_2(0,l)} \tag{19}$$

However, this doesn't solve my problem. In order to prove the formula for the Frechet gradient, I need ψ which satisfies (15). To establish the necessary integral identity (15), we apply an approximation argument. Let $\{g_n\}$ be a sequence of smooth function on $[0,l]$ such that

$$\|g_n - 2\left(u(x,T;v)-w(x)\right)\|_{L_2[0,l]} \to 0 \text{ as } n \to +\infty \tag{20}$$

let $\psi_n = \psi_n(x,t)$ be a solution of the problem (16)–(18) with (18) replaced by

$$\psi_n(x,T) = g_n(x),\ 0 \leq x \leq l \tag{18$'$}$$

There exists a unique classical solution ([1]) of the problem (16), (17), (??$'$) in $C_{x,t}^{2,1}(Q)$, and in particular $\psi_n \in W_2^{1,1}(Q)$, satisfies (??$'$), and the integral identity

$$\int_0^T \int_0^l \left(\eta\psi_{nt} - a^2\eta_x\psi_{nx}\right)\,dx\,dt - a^2\nu\int_0^T \psi_n(l,t)\eta(l,t)\,dt$$

for every $\eta \in W_2^{1,0}(Q)$. In particular, we can choose $\eta = \Delta u$ and get

$$\int_0^T \int_0^l \left(\Delta u\psi_{nt} - a^2\Delta u_x\psi_{nx}\right)\,dx\,dt - a^2\nu\int_0^T \psi_n(l,t)\Delta u(l,t)\,dt \tag{21}$$

The difference $\psi_n - \psi$ is $V_2^{1,0}(Q)$ solution of the the adjoint problem (16)–(18) with boundary function replaced by

$$g_n - 2\left(u(x,T;v)-w(x)\right)$$

By applying the energy estimate (19) to $\psi_n - \psi$ we have

$$\|\psi_n - \psi\|_{V_2^{1,0}(Q)} \leq C\left\|g_n - 2\left(u(x,T;v)-w(x)\right)\right\|_{L_2(0,l)} \to 0 \text{ as } n \to +\infty \tag{22}$$

Let us now choose in (11) (or (13)) ψ_n as a test function. Then we have

$$\begin{aligned}\int_0^l g_n(x)\Delta u(x,T)\,dx &= a^2\nu\int_0^T \psi_n(l,t)\Delta p(t)\,dt + \int_0^T\int_0^l \psi_n(x,t)\Delta f(x,t)\,dx\,dt + \\ &+ \int_0^T\int_0^l \left(\Delta u\psi_{nt} - a^2\Delta u_x\psi_{nx}\right)\,dx\,dt - a^2\nu\int_0^T \psi_n(l,t)\Delta u(l,t)\,dt = \\ &= a^2\nu\int_0^T \psi_n(l,t)\Delta p(t)\,dt + \int_0^T\int_0^l \psi_n(x,t)\Delta f(x,t)\,dx\,dt \end{aligned} \tag{23}$$

due to (21). Passing to the limit as $n \to +\infty$, and by using (20), (22) and trace embedding theorem we have

$$\int_0^l 2\left(u(x,T;v)-f(x)\right)\Delta u(x,T)\,dx = a^2\nu\int_0^T \psi(l,t)\Delta p(t)\,dt + \int_0^T\int_0^l \psi(x,t)\Delta f(x,t)\,dx\,dt$$

Hence, we have proved that

$$\Delta J(u) = a^2\nu\int_0^T \psi(l,t)\Delta p(t)\,dt + \int_0^T\int_0^l \psi(x,t)\Delta f(x,t)\,dx\,dt + \int_0^l \left|\Delta u(x,T)\right|^2\,dx \tag{24}$$

To complete the proof we have to show that

$$\int_0^l \left|\Delta u(x,T)\right|^2\,dx = o(\|\Delta v\|_H) \tag{25}$$

By using the $V_2^{1,0}(Q)$-energy estimate for Δu, we have

$$\int_0^l |\Delta u(x,T)|^2 \, dx \leq \|\Delta u\|^2_{V_2^{1,0}(Q)} \leq C(\|\Delta p\|^2_{L_2[0,T]} + \|\Delta f\|^2_{L_2(Q)}) = C \, \|\Delta v\|^2_H \tag{26}$$

From (26), (25) follows. But by using (25), from (24) it follows that the Frechet gradient exists and

$$J'(v) = \Big(a^2 \nu \psi(l,t), \psi(x,t)\Big) \in H \tag{27}$$

It is natural to call $a^2\nu\psi(l,t)$ a partial derivative of J with respect to $p \in L_2[0,T]$, and $\psi(x,t)$ a partial derivative of J with respect to $f \in L_2(Q)$.

References

[1] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, vol. 23 of Translations of Mathematical Monographs, American Mathematical Society, Providence, R. I., 1968.

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