LECTURE 1
INTRODUCTION TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS I. NONLINEAR DIFFUSION EQUATION

UGUR G. ABDULLA

I am going to start a series of lectures on nonlinear partial differential equations. Advancements made in the theory of nonlinear PDEs is one of the main achievements in XX century mathematics. Let me first make a remark on linear PDEs. In some sense one can say that there is a complete theory of linear PDEs, and perhaps the best source would be the four volumes of "The Analysis of Linear Partial Differential Operators" by Lars Hörmander [3]. Importance of the nonlinear PDEs is associated with the fact that they reflect more precise nonlinear laws of the nature, while linear PDEs were derived from the linearized versions of the nonlinear laws. Despite the great advances in the theory of nonlinear PDEs it is far from being complete. It is remarkable that almost every major nonlinear PDE has its own personality and requires unique approach. However, looking to current state of art in the theory of nonlinear PDEs, one can observe a similarity with the classical linear theory in the following sense: there are some major individual nonlinear PDEs, and analysis and understanding of those PDEs is a key towards the general theory of a broad class of nonlinear PDEs. I am going to concentrate on one of those key nonlinear PDEs called the nonlinear diffusion equation, which is a generalization of the classical heat/diffusion equation.

I will use the following notation

$$u = u(x,t),\ x \in \mathbb{R}^N,\ N \geq 1,\ t \in \mathbb{R}_+$$

$$\nabla u = \Big(u_{x_1}, \dots, u_{x_n}\Big), \Delta u = \sum_{i=1}^{N} u_{x_i x_i} = div(\nabla u)$$

The Nonlinear Diffusion Equation (NDE), or Porous Medium Equation (PME) can be written

$$u_t = \Delta\left(u^m\right),\ m > 0$$

When the solution can change sign, it takes the more general form

$$u_t = \Delta\left(|u|^{m-1}u\right)$$

We see that it is a diffusion type equation where the diffusion coefficient (or thermal conductivity, in the context of heat conduction) depends on temperature as a power-law. In particular, there is no diffusion where the concentration $u = 0$ when the exponent $n > 0$.

$$u_t = \operatorname{div}\left(u^n\,\nabla u\right)$$

When $m = 1$ (or $n = 0$ in the divergence form) this family of nonlinear diffusion equations includes the linear heat equation,

$$u_t = \Delta\, u$$

Note that the PME is a representative of a general class of nonlinear degenerate parabolic equations with implicit degeneracy.

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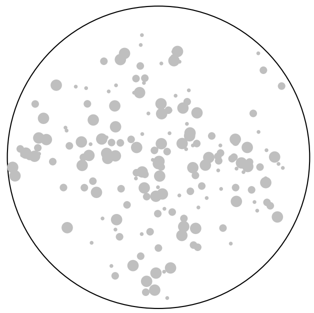
1. Physics

Consider the flow of gas through a porous medium: let ρ be density, $\vec{v}$ be velocity. If f is the porosity of the medium, $f \in (0,1]$. The microscopic picture is as below: gas is only able to flow where the empty region appears.



f then represents exactly the fraction of open space on this scale.
The governing equations in this case are

$$0 = f\rho_t + \operatorname{div}(\rho\vec{v}) \qquad \text{conservation of mass}$$

$$\vec{v} = -\frac{\kappa}{\mu}\nabla p \qquad \text{Darcy's law}$$

$$\rho = \rho_0 p^{\gamma} \qquad \text{equation of state}$$

where κ is the permeability and μ is the viscosity of the gas. We will eliminate $\vec{v}$ and p through these equations to derive a PDE for ρ; a nonlinear equation can also be derived for the pressure p.

$$\begin{aligned}
0 &= f\rho_t + \operatorname{div}\left(\rho\vec{v}\right)\\
&= f\rho_t + \operatorname{div}\left[\rho\left(-\frac{\kappa}{\mu}\nabla p\right)\right]\\
&= f\rho_t + \operatorname{div}\left[\rho\left(-\frac{\kappa}{\mu}\nabla\left(\frac{\rho}{\rho_0}\right)^{\frac{1}{\gamma}}\right)\right]
\end{aligned}$$

Calculate the gradient to get

$$\begin{aligned}
0 &= f\rho_t - \frac{\kappa}{\mu}\frac{1}{\rho_0^{\frac{1}{\gamma}}}\frac{1}{\gamma}\operatorname{div}\left[\rho^{\frac{1}{\gamma}}\nabla\rho\right]\\
&= f\rho_t - \frac{\kappa}{\mu}\frac{1}{\rho_0^{\frac{1}{\gamma}}}\frac{1}{\gamma}\frac{1}{1+\frac{1}{\gamma}}\operatorname{div}\nabla\rho^{\frac{1}{\gamma}+1}
\end{aligned}$$

Which can be written as

$$0 = \rho_t - c\,\Delta\rho^m$$

where $m = \frac{1}{\gamma} + 1$. By applying the time scaling $t \mapsto ct$, we can remove the constant and get the nonlinear diffusion equation:

$$\rho_t = \Delta\rho^m$$

In this derivation, one finds $m > 1$, but alternative derivations give a full range of exponents m. The family of nonlinear diffusion equations has a long history; Boussinesq derived this equation with $m = 2$ in 1903 [2].

2. Instantaneous Point-Source (ZKB) Solution

We would like to find a solution to the PME, in particular, inspired by the fundamental solution of the linear heat equation. Zel'dovich and Kompaneetz (1950), and Barenblatt (1952) both derived these type of solutions [1, 4].

Recall the instantaneous point-source problem for the heat equation,

$$u_t = \Delta u, \ x \in \mathbb{R}^N, \ t > 0 \tag{1}$$

$$u(x,0) = 0, \ x \neq 0 \tag{2}$$

$$\int_{\mathbb{R}^N} u(x,t)\, dx = 1, \ t \geq 0 \tag{3}$$

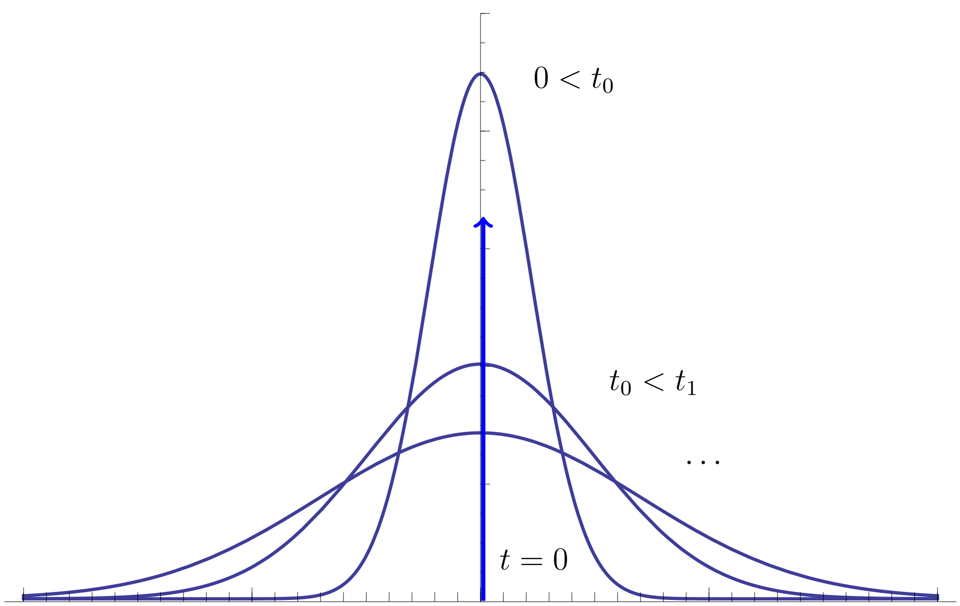
i.e.

$$u(x,0) = \delta(x)$$

having the so-called fundamental solution

$$\Phi(x,t) = (4\pi t)^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right)$$

and which looks as follows:



This solution is a key part of the representation formula for the Cauchy problem

$$u_t = \Delta u, \ x \in \mathbb{R}^N, \ t > 0$$

$$u(x,0) = \phi(x), \ x \in \mathbb{R}^N$$

which has the form

$$u(x,t) = \int_{\mathbb{R}^N} \Phi(x-y,t)\phi(y)\,dy \tag{4}$$

We will develop the solution to an analogous problem for the PME,

$$u_t - \operatorname{div}\Big(u^\sigma \nabla u\Big) = 0,\ x \in \mathbb{R}^N,\ t > 0 \tag{5}$$

$$\int_{\mathbb{R}^N} u(x,t)\,dx = 1,\ t \geq 0 \tag{6}$$

$$u(x,t) \geq 0 \tag{7}$$

Here the integral condition corresponds to conservation of energy, and $\sigma > 0$ is fixed.

Similar to the derivation of the fundamental solution to the heat equation, search solution in a self-similar form

$$u(x,t) = t^\alpha \theta(\xi), \quad \xi = \frac{x}{t^\beta},\ x \in \mathbb{R}^N$$

where α, β are to be determined. Calculate

$$\begin{aligned}
u_t &= \alpha t^{\alpha-1}\theta + t^\alpha \sum_{i=1}^N \theta_{\xi_i}\Big(\xi_i\Big)_t \\
&= \alpha t^{\alpha-1}\theta + t^\alpha \sum_{i=1}^N \theta_{\xi_i}(-\beta)t^{-\beta-1}x_i \\
&= \alpha t^{\alpha-1}\theta - \beta t^{\alpha-1}\xi \cdot \nabla \theta, \\
u_{x_i} &= t^\alpha \theta_{x_i} = t^{\alpha-\beta}\theta_{\xi_i} \\
u^\sigma u_{x_i} &= t^{\alpha\sigma}\theta^\sigma t^{\alpha-\beta}\theta_{\xi_i} \\
&= t^{\alpha(\sigma+1)-\beta}\theta^\sigma \theta_{\xi_i}
\end{aligned}$$

so

$$\begin{aligned}
\operatorname{div}\Big(u^\sigma \nabla u\Big) &= \sum_{i=1}^N \frac{\partial}{\partial x_i}\Big[t^{\alpha(\sigma+1)-\beta}\theta^\sigma \theta_{\xi_i}\Big] \\
&= \sum_{i=1}^N \frac{\partial}{\partial \xi_i}\Big[t^{\alpha(\sigma+1)-\beta}\theta^\sigma \theta_{\xi_i}\Big] t^{-\beta} \\
&= t^{\alpha(\sigma+1)-2\beta} \operatorname{div}\Big(\theta^\sigma \nabla \theta\Big)
\end{aligned}$$

Hence (5) implies

$$0 = u_t - \operatorname{div}\Big(u^\sigma \nabla u\Big) = \alpha t^{\alpha-1}\theta - \beta t^{\alpha-1}\xi \cdot \nabla \theta - t^{\alpha(\sigma+1)-2\beta} \operatorname{div}\Big(\theta^\sigma \nabla \theta\Big)$$

To eliminate t, choose $\alpha - 1 = \alpha(\sigma+1) - 2\beta$. Applying the conservation of energy fixes the coefficients α, β:

$$1 = \int_{\mathbb{R}^N} u(x,t)\,dx = t^\alpha \int_{\mathbb{R}^N} \theta(\xi) t^{N\beta}\,d\xi = t^{\alpha+N\beta} \int_{\mathbb{R}^N} \theta(\xi)\,d\xi$$

which is possible when $\alpha + N\beta = 0$. The solution of the system

$$\begin{cases} \sigma\alpha - 2\beta = -1 \\ \alpha + N\beta = 0 \end{cases}$$

gives

$$\alpha = \frac{-N}{2+N\sigma}, \quad \beta = \frac{1}{2+N\sigma}$$

We come to a problem for the shape function θ:

$$\begin{cases} \operatorname{div}\left(\theta^\sigma \nabla \theta\right) + \frac{N}{2+N\sigma}\theta + \frac{1}{2+N\sigma}\xi \cdot \nabla \theta, \quad \xi \in \mathbb{R}^N \\ \int_{\mathbb{R}^N} \theta(\xi)\, d\xi = 1 \end{cases} \tag{8}$$

From a physical standpoint, we expect the solution should be radially symmetric: $\theta = \theta(\eta)$, $\eta = |\xi| \geq 0$. Then

$$\theta_{\xi_i} = \theta'(\eta)\eta_{\xi_i} = \theta'(\eta)\frac{\xi_i}{\eta}$$
$$\xi \cdot \nabla \theta = \eta\theta'(\eta)$$

and

$$\begin{aligned} \operatorname{div}\left(\theta^\sigma \nabla \theta\right) &= \sum_{i=1}^N \frac{\partial}{\partial \xi_i}\left[\theta^\sigma\theta'\frac{\xi_i}{\eta}\right] \\ &= \sum_{i=1}^N \frac{\partial}{\partial \xi_i}\left(\theta^\sigma\theta'\right)\frac{\xi_i}{\eta} + \sum_{i=1}^N \theta^\sigma\theta'\frac{\partial}{\partial \xi_i}\left(\frac{\xi_i}{\eta}\right) \\ &= \sum_{i=1}^N \left(\theta^\sigma\theta'\right)'\frac{\xi_i^2}{\eta^2} + \sum_{i=1}^N \theta^\sigma\theta'\frac{\eta - \frac{\xi_i^2}{\eta}}{\eta^2} \\ &= \left(\theta^\sigma\theta'\right)' + \frac{N-1}{\eta}\theta^\sigma\theta' \end{aligned}$$

We rewrite this term as

$$\begin{aligned} \operatorname{div}\left(\theta^\sigma \nabla \theta\right) &= \frac{1}{\eta^{N-1}}\left[\eta^{N-1}\left(\theta^\sigma\theta'\right)' + (N-1)\eta^{N-2}\theta^\sigma\theta'\right] \\ &= \frac{1}{\eta^{N-1}}\left[\eta^{N-1}\theta^\sigma\theta'\right]' \end{aligned}$$

Plugging these to (8) gives

$$\frac{1}{\eta^{N-1}}\left[\eta^{N-1}\theta^\sigma\theta'\right]' + \frac{N}{2+N\sigma}\theta + \frac{1}{2+N\sigma}\eta\theta'(\eta) = 0$$
$$\left[\eta^{N-1}\theta^\sigma\theta'\right]' + \frac{1}{2+N\sigma}\left(\eta^N\theta\right)' = 0$$

Integrate and observe that the integration constant is zero since $\theta^\sigma\theta'(0) = 0$. The latter follows from the radial symmetricity and from the physical condition of the continuity of the flux $\theta^\sigma\theta'$.

$$\eta^{N-1}\theta^\sigma\theta' + \frac{1}{2+N\sigma}\eta^N\theta = 0$$

Hence

$$\theta^\sigma \theta' = -\frac{1}{2+N\sigma}\eta\theta$$
$$\theta^{\sigma-1}\theta' = -\frac{1}{2+N\sigma}\eta$$

Integrate to find

$$\frac{\theta^\sigma}{\sigma} = \frac{1}{2(2+N\sigma)}\left[\eta_0^2 - \eta^2\right],\ 0 \le \eta \le \eta_0$$

where η_0 is a constant of integration. Since θ^σ must be defined for all η, due to nonnegativity condition we are forced to continue by zero to whole space:

$$\frac{\theta^\sigma}{\sigma} = \frac{1}{2(2+N\sigma)}\left[\eta_0^2 - \eta^2\right]_+$$

where $(x)_+ = \max(x,0)$. Solving for θ gives the "shape function"

$$\theta(\eta) = \left[\frac{\sigma}{2(2+N\sigma)}\left(\eta_0^2 - \eta^2\right)_+\right]^{\frac{1}{\sigma}}$$

The constant η_0 is defined uniquely by the integral condition/conservation of mass in (8). Note that zero continuation caused discontinuity of the derivative θ' at $\eta = \eta_0$, and hence constructed solution is not a classical solution and one needs to clarify the notion of solution in order to be in agreement with mathematics. However, note that although θ' is discontinuous, the flux function $\theta^\sigma\theta'$ is everywhere continuous, which is in accordance with the conservation laws of physics.
Substituting $\eta = |\xi| = |x|/t^\beta$ gives the instantaneous point-source solution, also called the Zeldovich-Kompaneets-Barenblatt (ZKB) solution.

$$u_*(x,t) = t^{-\frac{N}{2+N\sigma}}\left[\frac{\sigma}{2(2+N\sigma)}\left(\eta_0^2 - \frac{|x|^2}{t^{\frac{2}{2+N\sigma}}}\right)_+\right]^{\frac{1}{\sigma}} \tag{9}$$

One can check directly that

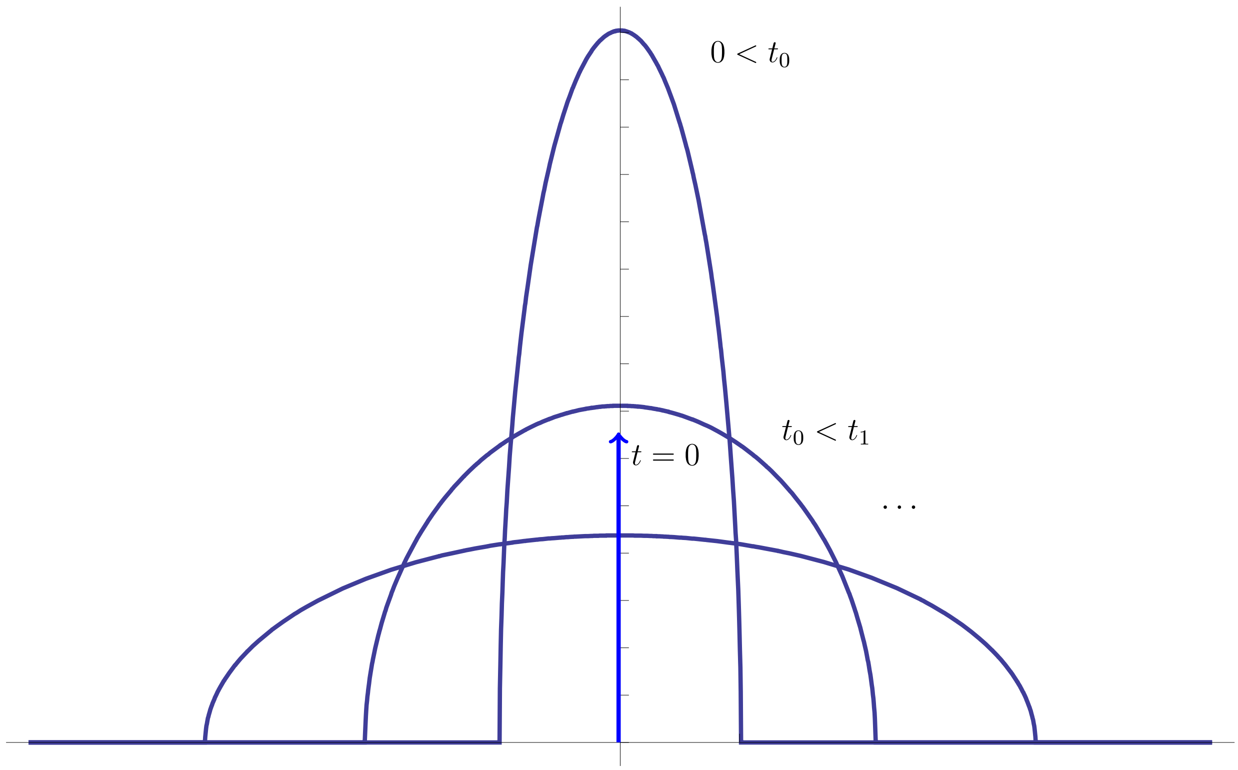
$$\lim_{t\downarrow 0} u(x,t) = \begin{cases} 0, \text{ if } x \neq 0, \\ +\infty, \text{ if } x = 0 \end{cases}$$

This together with conservation of mass condition (6) implies that in the sense of distributions

$$u(x,0) = \delta(x),\ x \in \mathbb{R}^N$$

where $\delta(\cdot)$ be a Dirac's point mass concentrated at the origin, or delta function.

Note that the shape function θ corresponds to $u(x,1)$! The solution evolves in time with the following shape:



3. Properties of the ZKB solution

Note: The limit of the u_* as $\sigma \downarrow 0$ is not the fundamental solution $\Phi(x, t)$ of the heat equation! There is some essential difference introduced by the nonlinear nature of the porous medium equation, even considering its close resemblance to the linear heat equation. Moreover, no representation formula involving the ZKB solution of a spirit similar to (4) seems to exist.

Interesting properties, meriting further study, include

(1) The ZKB solution exhibits a finite speed of propagation: with compactly supported initial data $u_*(x, t_0)$ (any $t_0 > 0$), the solution to the problem

$$\begin{cases} u_t = \operatorname{div}\Big(u^\sigma u\Big), & (x,t) \in \mathbb{R}^N \times \mathbb{R}_+ \\ u(x,0) = u_*(x,t_0), & x \in \mathbb{R}^N \end{cases}$$

is exactly $u_*(x, t + t_0)$ (the equation is invariant under translation of time or space coordinate) which has compact (i.e. bounded) support for all $t > 0$. Recall what we mean by support of a function:

$$\operatorname{spt}(u) = \overline{\{(x,t) : u(x,t) > 0\}}$$

For the ZKB solution,

$$\operatorname{spt}(u_*) = \Big\{(x,t) : |x| \leq \eta_0 t^{\frac{1}{N\sigma+2}}\Big\}$$

That is, the support, or positivity region, is given by a ball with a certain finite radius depending on time. We can also see this graphically in the plot of the solution; the finite speed of propagation property gives the Nonlinear Diffusion Equation what is called hyperbolic behavior characteristic for the linear wave equation.

This behavior is in stark contrast to the same problem for the linear heat equation: with any compactly supported initial data, the solution will become positive everywhere at any positive time. For this reason, it is said that the heat equation exhibits an infinite speed of propagation. From the standpoint of applications it is natural to expect that diffusion propogates in space with finite speed. Hence, nonlinear diffusion equation presents the phenomena which is more accurate reflection of the diffusion process in nature. That was a reason why starting from 1960s mathematical theory of nonlinear diffusion type equations became one of the central problems in the theory of nonlinear PDEs.

(2) The ZKB solution is not continuously differentiable: on the boundary of the support,

$$\left\{|x| = \eta_0 t^{\frac{1}{N\sigma+2}}\right\}$$

the gradient ∇u is discontinuous. Recall that we define a classical solution to the problem (5) as having a continuous derivative with respect to time and two continuous space derivatives: the solution we constructed does not have this property, exactly on the surface where the diffusion coefficient degenerates, $\{u = 0\}$.

In what sense, then, is $u_*(x,t)$ a solution to (5)–(7), if at all? Can there be a classical solution to the instantaneous point source problem, which would make ZKB solution some curiosity with no physical relevance? We will see more justification to this solution next time.

Here is my closing remark in favor of ZKB solution. Consider the way in which we define the classical solution: is there any requirement from nature that the gradient of temperature be continuous, as we do in the solution to the PDE? There don't seem to be any conservation laws which would require this in our context. However, the *heat flux* should be continuous, according to conservation of energy. The heat flux, the amount of heat flowing through a unit area per unit time, is proportional to

$$u^\sigma \nabla u$$

If you calculate this quantity for the ZKB solution, you will see it is continuous.

References

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[4] Ya. B. Zel'dovich and A. S. Kompaneets, *Towards a theory of heat condution with thermal conductivity depending on the temperature*, Izd. Akad. Nauk USSR Collection of Papers Dedicated to 70th Anniversary of A. F. Ioffe (1950), 61–72.

E-mail address: `abdulla@fit.edu`

Department of Mathematical Sciences, Florida Institute of Technology, 150 W. University Blvd., Melbourne, FL 32901