Topological Dynamics and Universality in Chaos
III. Proof of Sharkovski’s Theorem

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Theorem 1
(Sharkovsky, 1964) Let the positive integers be totally ordered in the following way:

\[ 1 \succ 2 \succ 2^2 \succ 2^3 \succ \ldots \succ 2^2 \cdot 5 \succ 2^2 \cdot 3 \succ \ldots \succ 2 \cdot 5 \succ 2 \cdot 3 \succ \ldots \succ 7 \succ 5 \succ 3 \]

If \( f \) has a cycle of period \( n \) and \( m \succ n \), then \( f \) also has a periodic orbit of period \( m \).

Lemma 2
If \( J \) is a compact subinterval such that \( J \subseteq f(J) \), then \( f \) has a fixed point in \( J \).

Lemma 3
If \( J, K \) are compact subintervals such that \( K \subseteq f(J) \), then there is a compact subinterval \( L \subseteq J \) such that \( f(L) = K \).
Lemma 4
If $J_0, J_1, \ldots, J_m$ are compact subintervals such that $J_k \subseteq f(J_{k-1})$ $(1 \leq k \leq m)$, then there is a compact subinterval $L \subseteq J_0$ such that $f^m(L) = J_m$ and $f^k(L) \subseteq J_k$ $(1 \leq k < m)$. If also $J_0 \subseteq J_m$, then there exists a point $y$ such that $f^m(y) = y$ and $f^k(y) \in J_k$ $(0 \leq k < m)$.

Lemma 5
Between any two points of a periodic orbit of period $n > 1$ there is a point of a periodic orbit of period less than $n$.

Let $B = \{x_1 < x_2 < \cdots < x_n\}$ be $n$-orbit of $f$.

Definition 1.1
If $f(x_i) = x_{s_i}, 1 \leq s_i \leq n, i = 1, 2, \ldots, n$, then $B$ is associated with cyclic permutation

$$
\begin{bmatrix}
1 & 2 & \ldots & n \\
S_1 & S_2 & \ldots & S_n
\end{bmatrix}
$$
Definition 1.2

Let $I_i = [x_i, x_{i+1}]$. Digraph of a cycle is a directed graph of transitions with vertices $I_1, I_2, \ldots, I_{n-1}$ and oriented edges $I_i \to I_s$ if $I_s \subseteq f(I_i)$.

Properties of Digraphs:

1. $\forall I_j \exists$ at least one $I_k$ for which $I_j \to I_k$. Moreover, it is always possible to choose $k \neq j$, unless $n = 2$.

2. $\forall I_k \exists$ at least one $I_j$ for which $I_j \to I_k$. Moreover, it is always possible to choose $j \neq k$, unless $n$ is even and $k = \frac{n}{2}$.

3. Digraph always contains a loop: $I_k \to I_k$. 
Definition 1.3
Given \( n \)-orbit, a cycle

\[ J_0 \rightarrow J_1 \rightarrow \cdots \rightarrow J_{n-1} \rightarrow J_0 \]

of length \( n \) in the digraph is called a **Fundamental Cycle** (FC) if \( J_0 \) contains an endpoint \( c \) s.t. \( f^k(c) \) is an endpoint of \( J_k \) for \( 1 \leq k < n \).

FC always exists and unique. In the FC some vertex must occur at least twice among \( J_0, \ldots, J_{n-1} \), since digraph has only \( n - 1 \) vertices. On the other hand, every vertex occurs at most twice, since interval \( I_k \) has two endpoints.

Definition 1.4
**Cycle in a digraph is said to be primitive if it does not consist entirely of a cycle of smaller length described several times.**

If FC contains \( I_k \) twice then it can be decomposed into two cycles of smaller length, each of which contains \( I_k \) once, and consequently is primitive.
Lemma 6
Suppose \( f \) has a periodic point of period \( n > 1 \). If the associated digraph contains a primitive cycle

\[
J_0 \to J_1 \to \cdots \to J_{m-1} \to J_0
\]

of length \( m \), then \( f \) has a periodic point \( y \) of period \( m \) such that \( f^k(y) \in J_k (0 \leq k < m) \).

Suppose \( f \) has a 3-orbit: \( f(c) < c < f^2(c) \) with corresponding digraph

\[
\circlearrowleft I_1 \Leftrightarrow I_2
\]

\( I_1 \to I_1 \Rightarrow \) there is a fixed point; \( I_1 \to I_2 \to I_1 \Rightarrow \) there is a 2-orbit

\( \forall \) positive integer \( m > 2 \) there is an \( m \)-orbit corresponding to primitive cycle of length \( m \): \( I_1 \to I_2 \to I_1 \to I_1 \to \cdots \to I_1 \)

Lemma 7
If \( f \) has a periodic point of period \( > 1 \), then it has a fixed point and a periodic point of period 2.
Lemma 8
Suppose \( f \) has a periodic orbit of odd period \( n > 1 \), but no periodic orbit of odd period strictly between 1 and \( n \). If \( c \) is the midpoint of the orbit of odd period \( n \), then the points of this orbit have the order

\[
\begin{align*}
  f^{n-1}(c) &< f^{n-3}(c) < \cdots < f^2(c) < c < f(c) < \cdots < f^{n-2}(c) \\
\end{align*}
\]

or the inverse order

\[
\begin{align*}
  f^{n-2}(c) &< \cdots < f(c) < c < f^2(c) < \cdots < f^{n-3}(c) < f^{n-1}(c) \\
\end{align*}
\]

and associated digraph is given in the figure, where \( J_1 = < c, f(c) > \) and \( J_k = < f^{k-2}(c), f^k(c) > \) for \( 1 < k < n \).
Lemma 9
If \( f \) has a periodic orbit of odd period \( n > 1 \), then it has periodic points of arbitrary even order and periodic points of arbitrary odd order > \( n \).

Proof. We may assume \( n \) is minimal. Then digraph is a Stefan graph as in Lemma 8. If \( m < n \) is even then

\[
J_{n-1} \to J_{n-m} \to J_{n-m+1} \to \cdots \to J_{n-1}
\]

is a primitive cycle of length \( m \). If \( m > n \) is even or odd then

\[
J_1 \to J_2 \to \cdots \to J_{n-1} \to J_1 \to J_1 \to \cdots \to J_1
\]

is a primitive cycle of length \( m \). \( \square \)
Proof of Sharkovski’s Theorem

Lemma 10
If \( c \) is a periodic point of \( f \) with period \( n \) then for any positive integer \( h \), \( c \) is a periodic point of \( f^h \) with period \( \frac{n}{(h,n)} \), where \( (h,n) \) denotes the greatest common divisor of \( h \) and \( n \).
Conversely, if \( c \) is a periodic point of \( f^h \) with period \( m \) then \( c \) is a periodic point of \( f \) with period \( \frac{mh}{d} \), where \( d \) divides \( h \) and is relatively prime to \( m \).
Lemma 10

If $c$ is a periodic point of $f$ with period $n$ then for any positive integer $h$, $c$ is a periodic point of $f^h$ with period $\frac{n}{(h,n)}$, where $(h,n)$ denotes the greatest common divisor of $h$ and $n$.

Conversely, if $c$ is a periodic point of $f^h$ with period $m$ then $c$ is a periodic point of $f$ with period $\frac{mh}{d}$, where $d$ divides $h$ and is relatively prime to $m$.

Proof. Suppose $c$ has period $n$ for $f$ and let $m = \frac{n}{(h,n)}$. We have

$$f^{mh}(c) = f^{\frac{nh}{(h,n)}}(c) = c$$

On the other hand, if $f^{kh}(c) = c$ then $n$ must be a factor of $kh$, say $kh = dn$. This implies that $m$ is a factor of $k$. Indeed

$$k = \frac{dn}{h} = \frac{n}{(h,n)} \frac{d(h,n)}{h} = m \frac{dh}{h} \frac{dn}{h} = m \frac{dh}{h} \frac{kh}{h} = m(d,k)$$

Hence, $c$ is $m$-periodic point for $f^h$ and first assertion is proved.
Suppose now that $c$ has period $m$ for $f^h$. Then $c$ has a period $n$ for $f$ where $n$ is a factor of $mh$, say $n = \frac{mh}{d}$. From the first assertion of lemma it follows that

$$m = \frac{n}{(h, n)} = \frac{nd}{h} \Rightarrow h = d(h, n) = de$$

and

$$(de, me) = (h, m(h, n)) = (h, n) = e \Rightarrow (d, m) = 1$$
Let $n = 2^d q$, where $q$ is odd. First assume $q = 1$ and $m = 2^e$, where $0 \leq e < d$. By Lemma 7 we may assume $e > 0$. Prove that $m \succ n$.

Consider a map $g = f^{m/2}$ and apply first assertion of the Lemma 10 with $h = m/2 = 2^{e-1}$, $n = 2^d$. It follows that $g$ has a periodic point $c$ of period

$$
\frac{n}{(h, n)} = \frac{2^d}{(2^{e-1}, 2^d)} = 2^{d-e+1}
$$

Lemma 7 $\Rightarrow$ $g$ has a periodic point of period 2. Apply second part of Lemma 10 with $h = m/2 = 2^{e-1}$ and $m = 2$: periodic point of $f^{2^{e-1}}$ with period 2, is a periodic point of $f$ with the period

$$
\frac{2 \cdot 2^{e-1}}{d} = \frac{2^e}{d},
$$

where $d$ is a factor of $2^{e-1}$ which is relatively prime with 2. Hence, $d = 1$, and $f$ has a periodic point of period $m = 2^e$. 
Now let \( n = 2^d q \), where \( q > 1 \) is odd. It remains to prove \( m \succ n \) for \( m = 2^d r \), where either (i) \( r \) is even, or (ii) \( r \) is odd and \( r > q \). Consider a map \( g = f^{2^d} \). Apply first part of Lemma 10 with \( h = 2^d \) and \( n = 2^d q \). It follows that \( g \) has a periodic point of period

\[
\frac{n}{(h, n)} = \frac{2^d q}{(2^d, 2^d q)} = \frac{2^d q}{2^d} = q.
\]

Lemma 9 \( \Rightarrow \) \( g \) has a periodic point of period \( r \). Now apply second assertion of the Lemma 10 with \( h = 2^d \) and \( m = r \). This point is a periodic point for \( f \) with the period \( mh/\bar{d} = r2^d/\bar{d} \), where \( \bar{d} \) divides \( 2^d \) and relatively prime to \( r \). In case (i) \( \bar{d} = 1 \) and \( f \) has a periodic point of period \( 2^d r \) as required. In case (ii) \( \bar{d} \) is some power of 2, and \( f \) has periodic point of period \( 2^e r \) for some \( e \leq d \). If \( e = d \) then we are done. If \( e < d \) we can replace \( n \) by \( 2^e r \). Since \( m = 2^e (2^d - e r) \) it then follows from the case (i) that \( f \) also has a periodic point of period \( m \). \qed