Topological Dynamics and Universality in Chaos
III. Proof of Sharkovski's Theorem

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Sharkovski's Theorem

Theorem 1

(Sharkovsky, 1964) Let the positive integers be totally ordered in the following way:

$$1 \succ 2 \succ 2^2 \succ 2^3 \succ ... \succ 2^2{\cdot}5 \succ 2^2{\cdot}3 \succ ... \succ 2{\cdot}5 \succ 2{\cdot}3 \succ ... \succ 7 \succ 5 \succ 3$$

If f has a cycle of period n and $m \succ n$, then f also has a periodic orbit of period m.

Lemma 2

If J is a compact subinterval such that $J \subseteq f(J)$, then f has a fixed point in J.

Lemma 3

If J, K are compact subintervals such that $K \subseteq f(J)$, then there is a compact subinterval $L \subseteq J$ such that $f(L) = K$.

Proof of Sharkovski's Theorem

Lemma 4

If $J_0, J_1, ..., J_m$ are compact subintervals such that $J_k \subseteq f(J_{k-1})$ $(1 \leq k \leq m)$, then there is a compact subinterval $L \subseteq J_0$ such that $f^m(L) = J_m$ and $f^k(L) \subseteq J_k$ $(1 \leq k < m)$.
If also $J_0 \subseteq J_m$, then there exists a point y such that $f^m(y) = y$ and $f^k(y) \in J_k$ $(0 \leq k < m)$.

Lemma 5

Between any two points of a periodic orbit of period $n > 1$ there is a point of a periodic orbit of period less than n.

Let $\mathbf{B} = \{x_1 < x_2 < \cdots < x_n\}$ be n-orbit of f.

Definition 1.1

If $f(x_i) = x_{s_i}, 1 \leq s_i \leq n, i = 1, 2, ..., n$, then $\mathbf{B}$ is associated with cyclic permutation

$$\begin{bmatrix} 1 & 2 & \dots & n \\ s_1 & s_2 & \dots & s_n \end{bmatrix}$$

Definition 1.2

Let $I_i = [x_i, x_{i+1}]$*. Digraph of a cycle is a directed graph of transitions with vertices* $I_1, I_2, \cdots, I_{n-1}$ *and oriented edges* $I_i \to I_s$ *if* $I_s \subseteq f(I_i)$*.*

Properties of Digraphs:

1. $\forall\ I_j\ \exists$ at least one I_k for which $I_j \to I_k$. Moreover, it is always possible to choose $k \neq j$, unless $n = 2$.
2. $\forall\ I_k\ \exists$ at least one I_j for which $I_j \to I_k$. Moreover, it is always possible to choose $j \neq k$, unless n is even and $k = \frac{n}{2}$.
3. Digraph always contains a loop: $I_k \to I_k$.

Definition 1.3

Given n-orbit, a cycle

$$J_0 \to J_1 \to \cdots \to J_{n-1} \to J_0$$

of length n in the digraph is called a **Fundamental Cycle** *(FC) if J_0 contains an endpoint c s.t. $f^k(c)$ is an endpoint of J_k for $1 \leq k < n$.*

FC always exists and unique. In the FC some vertex must occur at least twice among $J_0, ..., J_{n-1}$, since digraph has only $n-1$ vertices. On the other hand, every vertex occurs at most twice, since interval I_k has two endpoints.

Definition 1.4

Cycle in a digraph is said to be primitive if it does not consisit entirely of a cycle of smaller length described several times.

If FC contains I_k twice then it can be decomposed into two cycles of smaller length, each of which contains I_k once, and consequently is primitive.

Lemma 6

Suppose f has a periodic point of period $n > 1$. If the associated digraph contains a primitive cycle

$$J_0 \to J_1 \to \cdots \to J_{m-1} \to J_0$$

of length m, then f has a periodic point y of period m such that $f^k(y) \in J_k (0 \leq k < m)$.

Suppose f has a 3-orbit: $f(c) < c < f^2(c)$ with corresponding digraph

$$\circlearrowright I_1 \rightleftarrows I_2$$

$I_1 \to I_1 \Rightarrow$ there is a fixed point; $I_1 \to I_2 \to I_1 \Rightarrow$ there is a 2-orbit
$\forall$ positive integer $m > 2$ there is an m-orbit corresponding to primitive cycle of length m: $I_1 \to I_2 \to I_1 \to I_1 \to \cdots \to I_1$

Lemma 7

If f has a periodic point of period > 1, then it has a fixed point and a periodic point of period 2.

Lemma 8

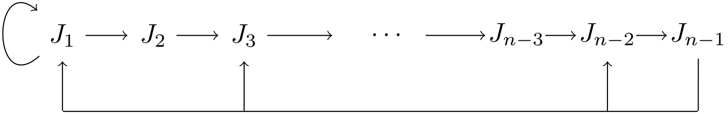
Suppose f has a periodic orbit of odd period $n > 1$, but no periodic orbit of odd period strictly between 1 and n. If c is the midpoint of the orbit of odd period n, then the points of this orbit have the order

$$f^{n-1}(c) < f^{n-3}(c) < \cdots < f^2(c) < c < f(c) < \cdots < f^{n-2}(c)$$

or the inverse order

$$f^{n-2}(c) < \cdots < f(c) < c < f^2(c) < \cdots < f^{n-3}(c) < f^{n-1}(c)$$

and associated digraph is given in the figure, where $J_1 =< c, f(c) >$ and $J_k =< f^{k-2}(c), f^k(c) >$ for $1 < k < n$.



Proof of Sharkovski's Theorem

Lemma 9

If f has a periodic orbit of odd period $n > 1$, then it has periodic points of arbitrary even order and periodic points of arbitrary odd order $> n$.

Proof. We may assume n is minimal. Then digraph is a Stefan graph as in Lemma 8. If $m < n$ is even then

$$J_{n-1} \to J_{n-m} \to J_{n-m+1} \to \cdots \to J_{n-1}$$

is a primitive cycle of length m. If $m > n$ is even or odd then

$$J_1 \to J_2 \to \cdots \to J_{n-1} \to J_1 \to J_1 \to \cdots \to J_1$$

is a primitive cycle of length m. □

Proof of Sharkovski's Theorem

Lemma 10

If c is a periodic point of f with period n then for any positive integer h, c is a periodic point of f^h with period $\frac{n}{(h,n)}$, where (h,n) denotes the greatest common divisor of h and n.
Conversely, if c is a periodic point of f^h with period m then c is a periodic point of f with period $\frac{mh}{d}$, where d divides h and is relatively prime to m.

Proof. Suppose c has period n for f and let $m = \frac{n}{(h,n)}$. We have

$$f^{mh}(c) = f^{\frac{nh}{(h,n)}}(c) = c$$

On the other hand, if $f^{kh}(c) = c$ then n must be a factor of kh, say $kh = dn$. This implies that m is a factor of k. Indeed

$$k = \frac{dn}{h} = \frac{n}{(h,n)} \frac{d(h,n)}{h} = m\frac{(dh,dn)}{h} = m\frac{(dh,kh)}{h} = m(d,k)$$

Hence, c is m-periodic point for f^h and first assertion is proved.

Suppose now that c has period m for f^h. Then c has a period n for f where n is a factor of mh, say $n = \frac{mh}{d}$. From the first assertion of lemma it follows that

$$m = \frac{n}{(h,n)} = \frac{nd}{h} \Rightarrow h = d(h,n) = de$$

and

$$(de, me) = (h, m(h,n)) = (h,n) = e \Rightarrow (d,m) = 1 \quad \square$$

Proof of Sharkovski's Theorem

Let $n = 2^d q$, where q is odd. First assume $q = 1$ and $m = 2^e$, where $0 \leq e < d$. By Lemma 7 we may assume $e > 0$. Prove that $m \succ n$. Consider a map $g = f^{\frac{m}{2}}$ and apply first assertion of the Lemma 10 with $h = \frac{m}{2} = 2^{e-1}$, $n = 2^d$. It follows that g has a periodic point c of period

$$\frac{n}{(h,n)} = \frac{2^d}{(2^{e-1}, 2^d)} = 2^{d-e+1}$$

Lemma 7 $\Rightarrow$ g has a periodic point of period 2. Apply second part of Lemma 10 with $h = \frac{m}{2} = 2^{e-1}$ and $m = 2$: periodic point of $f^{2^{e-1}}$ with period 2, is a periodic point of f with the period

$$\frac{2 \cdot 2^{e-1}}{d} = \frac{2^e}{d},$$

where d is afactor of 2^{e-1} which is relatively prime with 2. Hence, $d = 1$, and f has a periodic point of period $m = 2^e$.

Proof of Sharkovski's Theorem

Now let $n = 2^d q$, where $q > 1$ is odd. It remains to prove $m \succ n$ for $m = 2^d r$, where either (i) r is even, or (ii) r is odd and $r > q$. Consider a map $g = f^{2^d}$. Apply first part of Lemma 10 with $h = 2^d$ and $n = 2^d q$. It follows that g has a periodic point of period

$$\frac{n}{(h,n)} = \frac{2^d q}{(2^d, 2^d q)} = \frac{2^d q}{2^d} = q.$$

Lemma 9 $\Rightarrow$ g has a periodic point of period r. Now apply second assertion of the Lemma 10 with $h = 2^d$ and $m = r$. This point is a periodic point for f with the period $mh/\bar{d} = r2^d/\bar{d}$, where $\bar{d}$ divides 2^d and relatively prime to r. In case (i) $\bar{d} = 1$ and f has a periodic point of period $2^d r$ as required. In case (ii) $\bar{d}$ is some power of 2, and f has periodic point of period $2^e r$ for some $e \leq d$. If $e = d$ then we are done. If $e < d$ we can replace n by $2^e r$. Since $m = 2^e(2^{d-e} r)$ it then follows from the case (i) that f also has a periodic point of period m. □