Topological Dynamics and Universality in Chaos II. Proof of Sharkovski's Theorem

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Sharkovski's Theorem

Theorem 1

(Sharkovsky, 1964) Let the positive integers be totally ordered in the following way:

$$1 \succ 2 \succ 2^2 \succ 2^3 \succ ... \succ 2^2{\cdot}5 \succ 2^2{\cdot}3 \succ ... \succ 2{\cdot}5 \succ 2{\cdot}3 \succ ... \succ 7 \succ 5 \succ 3$$

If f has a cycle of period n and $m \succ n$, then f also has a periodic orbit of period m.

Lemma 2

If J is a compact subinterval such that $J \subseteq f(J)$, then f has a fixed point in J.

Lemma 3

If J, K are compact subintervals such that $K \subseteq f(J)$, then there is a compact subinterval $L \subseteq J$ such that $f(L) = K$.

Proof of Sharkovski's Theorem

Lemma 4

If $J_0, J_1, ..., J_m$ *are compact subintervals such that* $J_k \subseteq f(J_{k-1})$ $(1 \leq k \leq m)$*, then there is a compact subinterval* $L \subseteq J_0$ *such that* $f^m(L) = J_m$ *and* $f^k(L) \subseteq J_k$ $(1 \leq k < m)$*.*
If also $J_0 \subseteq J_m$*, then there exists a point* y *such that* $f^m(y) = y$ *and* $f^k(y) \in J_k$ $(0 \leq k < m)$*.*

Lemma 5

Between any two points of a periodic orbit of period $n > 1$ *there is a point of a periodic orbit of period less than* n*.*

Let $\mathbf{B} = \{x_1 < x_2 < \cdots < x_n\}$ be n-orbit of f.

Definition 1.1

If $f(x_i) = x_{s_i}, 1 \leq s_i \leq n, i = 1, 2, ..., n$*, then* $\mathbf{B}$ *is associated with cyclic permutation*

$$\begin{bmatrix} 1 & 2 & \dots & n \\ s_1 & s_2 & \dots & s_n \end{bmatrix}$$

Proof of Sharkovski's Theorem

Definition 1.2

Let $I_i = [x_i, x_{i+1}]$. Digraph of a cycle is a directed graph of transitions with vertices $I_1, I_2, \cdots, I_{n-1}$ and oriented edges $I_i \to I_s$ if $I_s \subseteq f(I_i)$.

Properties of Digraphs:

1. $\forall\ I_j\ \exists$ at least one I_k for which $I_j \to I_k$. Moreover, it is always possible to choose $k \neq j$, unless $n = 2$.
2. $\forall\ I_k\ \exists$ at least one I_j for which $I_j \to I_k$. Moreover, it is always possible to choose $j \neq k$, unless n is even and $k = \frac{n}{2}$.
3. Digraph always contains a loop: $I_k \to I_k$.

Definition 1.3

Given n-orbit, a cycle

$$J_0 \to J_1 \to \cdots \to J_{n-1} \to J_0$$

of length n in the digraph is called a **Fundamental Cycle** *(FC) if J_0 contains an endpoint c s.t. $f^k(c)$ is an endpoint of J_k for $1 \leq k < n$.*

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FC always exists and unique. In the FC some vertex must occur at least twice among $J_0, ..., J_{n-1}$, since digraph has only $n-1$ vertices. On the other hand, every vertex occurs at most twice, since interval I_k has two endpoints.

Definition 1.4

Cycle in a digraph is said to be primitive if it does not consisit entirely of a cycle of smaller length described several times.

If FC contains I_k twice then it can be decomposed into two cycles of smaller length, each of which contains I_k once, and consequently is primitive.

Lemma 6

Suppose f has a periodic point of period $n > 1$. If the associated digraph contains a primitive cycle

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of length m, then f has a periodic point y of period m such that $f^k(y) \in J_k (0 \leq k < m)$.

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Proof. $J_1 \subseteq f(J_0), J_2 \subseteq f(J_1), \cdots, J_{m-1} \subseteq f(J_{m-2}), J_0 \subseteq f(J_{m-1})$

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Either m is a period of y or period of y is a factor of m. If y is not an endpoint of J_0, then m is a period of y since cycle is primitive. Assume y is an endpoint of J_0. Since y is an element of n-orbit $\Rightarrow$ n is a divisor of m. We have $J_k \subseteq f(J_{k-1})$ and $f^k(y) \in J_k \Rightarrow J_k$ is defined uniquely and moreover, cycle is a multiple of the FC. This is a contradiction, unless $n = m$, since cycle is primitive. □

Suppose f has a 3-orbit: $f(c) < c < f^2(c)$ with corresponding digraph

$$\circlearrowright I_1 \rightleftarrows I_2$$

Proof of Sharkovski's Theorem

Straffin's Lemma $\Rightarrow 3 \prec m \prec 2 \prec 1$

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$\forall$ positive integer $m > 2$ there is an m-orbit corresponding to primitive cycle of length m: $I_1 \to I_2 \to I_1 \to I_1 \to \cdots \to I_1$

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If f has a periodic point of period > 1, then it has a fixed point and a periodic point of period 2.

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Lemma 7

If f has a periodic point of period > 1, then it has a fixed point and a periodic point of period 2.

Proof. Digraph has a loop $\Rightarrow$ there is a fixed point. Let $n > 1$ be the least positive integer such that f has a periodic point of period n. If $n > 2$ decompose FC into two primitive cycles. Since at least one of these has length greater than 1, by Straffin's lemma we deduce there is a periodic point of period strictly between 1 and n. □

Lemma 8

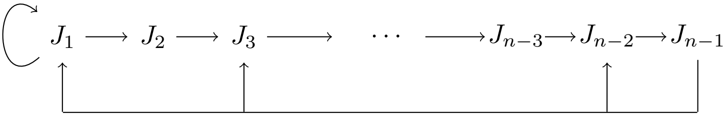
Suppose f has a periodic orbit of odd period $n > 1$, but no periodic orbit of odd period strictly between 1 and n. If c is the midpoint of the orbit of odd period n, then the points of this orbit have the order

$$f^{n-1}(c) < f^{n-3}(c) < \cdots < f^2(c) < c < f(c) < \cdots < f^{n-2}(c)$$

or the inverse order

$$f^{n-2}(c) < \cdots < f(c) < c < f^2(c) < \cdots < f^{n-3}(c) < f^{n-1}(c)$$

and associated digraph is given in the figure, where $J_1 =< c, f(c) >$ and $J_k =< f^{k-2}(c), f^k(c) >$ for $1 < k < n$.



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where $J_i \neq J_1$ for $1 < i < n$. If $J_i = J_k$, $1 < i < k < n$, then we obtain a smaller primitive cycle, and by excluding the loop at J_1 if necessary, we can arrange that its length is odd. Hence, $J_1, ..., J_{n-1}$ are all distinct and thus a permutation of $I_1, ..., I_{n-1}$. Similarly, we cannot have $J_i \to J_k$ if $k > i + 1$ or if $k = 1$ and $i \neq 1, n - 1$.

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$$x_k = a,\ x_{k+1} = f(a),\ x_{k-1} = f^2(a)$$

or

$$x_{k+1} = b,\ x_k = f(b),\ x_{k+2} = f^2(b).$$

Consider the first case, the argument in the second being similar. If $f^3(a) < f^2(a)$ then $J_2 \to J_1$, which is forbidden. Hence $f^3(a) > f^2(a)$.

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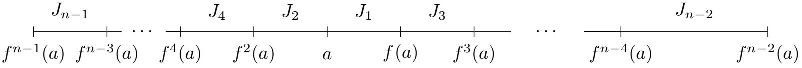

Since the endpoints of $J_{n-1} = [x_1, x_2]$ are mapped into a and $f^{n-2}(a) = x_n$ we have $J_{n-1} \to J_k$ iff k is odd. We found all the arcs in the digraph. □

Proof of Sharkovski's Theorem

Lemma 9

If f has a periodic orbit of odd period $n > 1$, then it has periodic points of arbitrary even order and periodic points of arbitrary odd order $> n$.

Proof. We may assume n is minimal. Then digraph is a Stefan graph as in Lemma 8. If $m < n$ is even then

$$J_{n-1} \to J_{n-m} \to J_{n-m+1} \to \cdots \to J_{n-1}$$

is a primitive cycle of length m. If $m > n$ is even or odd then

$$J_1 \to J_2 \to \cdots \to J_{n-1} \to J_1 \to J_1 \to \cdots \to J_1$$

is a primitive cycle of length m. □