# Topological Dynamics and Universality in Chaos II. Proof of Sharkovski's Theorem

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FIT Colloquium

June 2, 2014

# Sharkovski's Theorem

## Theorem 1

*(Sharkovsky, 1964) Let the positive integers be totally ordered in the following way:*

$$1 \succ 2 \succ 2^2 \succ 2^3 \succ ... \succ 2^2 \cdot 5 \succ 2^2 \cdot 3 \succ ... \succ 2 \cdot 5 \succ 2 \cdot 3 \succ ... \succ 7 \succ 5 \succ 3$$

*If $f$ has a cycle of period $n$ and $m \succ n$, then $f$ also has a periodic orbit of period $m$.*

## Lemma 2

*If $J$ is a compact subinterval such that $J \subseteq f(J)$, then $f$ has a fixed point in $J$.*

## Lemma 3

*If $J, K$ are compact subintervals such that $K \subseteq f(J)$, then there is a compact subinterval $L \subseteq J$ such that $f(L) = K$.*

# Proof of Sharkovski's Theorem

## Lemma 4

*If $J_0, J_1, ..., J_m$ are compact subintervals such that $J_k \subseteq f(J_{k-1})$ $(1 \leq k \leq m)$, then there is a compact subinterval $L \subseteq J_0$ such that $f^m(L) = J_m$ and $f^k(L) \subseteq J_k$ $(1 \leq k < m)$.*
*If also $J_0 \subseteq J_m$, then there exists a point $y$ such that $f^m(y) = y$ and $f^k(y) \in J_k$ $(0 \leq k < m)$.*

## Lemma 5

*Between any two points of a periodic orbit of period $n > 1$ there is a point of a periodic orbit of period less than $n$.*

Let $\mathbf{B} = \{x_1 < x_2 < \cdots < x_n\}$ be $n$-orbit of $f$.

## Definition 1.1

*If $f(x_i) = x_{s_i}, 1 \leq s_i \leq n, i = 1, 2, ..., n$, then $\mathbf{B}$ is associated with cyclic permutation*

$$\begin{bmatrix} 1 & 2 & \dots & n \\ s_1 & s_2 & \dots & s_n \end{bmatrix}$$

# Proof of Sharkovski's Theorem

## Definition 1.2

*Let $I_i = [x_i, x_{i+1}]$. Digraph of a cycle is a directed graph of transitions with vertices $I_1, I_2, \cdots, I_{n-1}$ and oriented edges $I_i \to I_s$ if $I_s \subseteq f(I_i)$.*

**Properties of Digraphs:**

1. $\forall\ I_j\ \exists$ at least one $I_k$ for which $I_j \to I_k$. Moreover, it is always possible to choose $k \neq j$ , unless $n = 2$.
2. $\forall\ I_k\ \exists$ at least one $I_j$ for which $I_j \to I_k$. Moreover, it is always possible to choose $j \neq k$, unless $n$ is even and $k = \frac{n}{2}$.
3. Digraph always contains a loop: $I_k \to I_k$.

## Definition 1.3

*Given $n$-orbit, a cycle*

$$J_0 \to J_1 \to \cdots \to J_{n-1} \to J_0$$

*of length $n$ in the digraph is called a* **Fundamental Cycle** *(FC) if $J_0$ contains an endpoint $c$ s.t. $f^k(c)$ is an endpoint of $J_k$ for $1 \leq k < n$.*

FC always exists and unique. In the FC some vertex must occur at least twice among $J_0, ..., J_{n-1}$, since digraph has only $n-1$ vertices. On the other hand, every vertex occurs at most twice, since interval $I_k$ has two endpoints.

## Definition 1.4

*Cycle in a digraph is said to be primitive if it does not consisit entirely of a cycle of smaller length described several times.*

If FC contains $I_k$ twice then it can be decomposed into two cycles of smaller length, each of which contains $I_k$ once, and consequently is primitive.

## Lemma 6

*Suppose $f$ has a periodic point of period $n > 1$. If the associated digraph contains a primitive cycle*

$$J_0 \to J_1 \to \cdots \to J_{m-1} \to J_0$$

*of length $m$, then $f$ has a periodic point $y$ of period $m$ such that $f^k(y) \in J_k (0 \leq k < m)$.*

Proof. $J_1 \subseteq f(J_0), J_2 \subseteq f(J_1), \cdots, J_{m-1} \subseteq f(J_{m-2}), J_0 \subseteq f(J_{m-1})$ Lemma 4 (w. $J_m = J_0$) $\Rightarrow \exists\, y \in J_0\ f^m(y) = y,\ f^k(y) \in J_k (0 \leq k < m)$ Either $m$ is a period of $y$ or period of $y$ is a factor of $m$. If $y$ is not an endpoint of $J_0$, then $m$ is a period of $y$ since cycle is primitive. Assume $y$ is an endpoint of $J_0$. Since $y$ is an element of $n$-orbit $\Rightarrow n$ is a divisor of $m$. We have $J_k \subseteq f(J_{k-1})$ and $f^k(y) \in J_k \Rightarrow J_k$ is defined uniquely and moreover, cycle is a multiple of the FC. This is a contradiction, unless $n = m$, since cycle is primitive. □

# Proof of Sharkovski's Theorem

**Straffin's Lemma** $\Rightarrow 3 \prec m \prec 2 \prec 1$

Suppose $f$ has a 3-orbit: $f(c) < c < f^2(c)$ with corresponding digraph

$$\circlearrowright I_1 \rightleftarrows I_2$$

$I_1 \to I_1 \Rightarrow$ there is a fixed point; $I_1 \to I_2 \to I_1 \Rightarrow$ there is a 2-orbit
$\forall$ positive integer $m > 2$ there is an $m$-orbit corresponding to primitive cycle of length $m$: $I_1 \to I_2 \to I_1 \to I_1 \to \cdots \to I_1$

## Lemma 7

*If $f$ has a periodic point of period $> 1$, then it has a fixed point and a periodic point of period 2.*

Proof. Digraph has a loop $\Rightarrow$ there is a fixed point. Let $n > 1$ be the least positive integer such that $f$ has a periodic point of period $n$. If $n > 2$ decompose FC into two primitive cycles. Since at least one of these has length greater than 1, by Straffin's lemma we deduce there is a periodic point of period strictly between 1 and $n$. $\square$

# Proof of Sharkovski's Theorem

**Stefan Orbits**

## Lemma 8

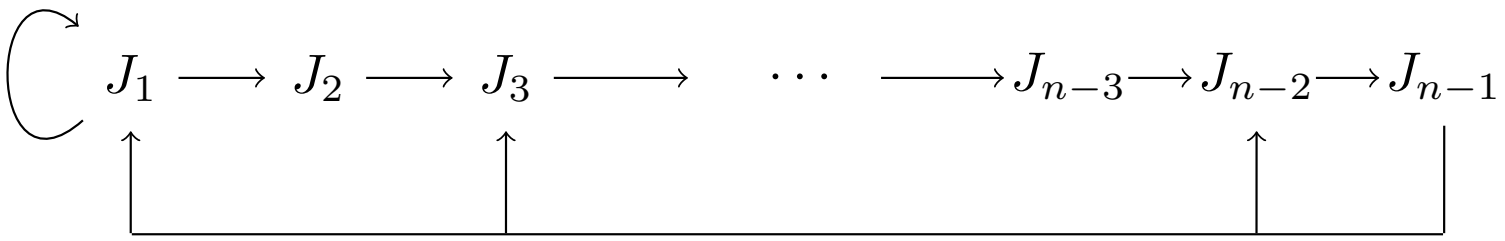
*Suppose $f$ has a periodic orbit of odd period $n > 1$, but no periodic orbit of odd period strictly between 1 and $n$. If $c$ is the midpoint of the orbit of odd period $n$, then the points of this orbit have the order*

$$f^{n-1}(c) < f^{n-3}(c) < \cdots < f^2(c) < c < f(c) < \cdots < f^{n-2}(c)$$

*or the inverse order*

$$f^{n-2}(c) < \cdots < f(c) < c < f^2(c) < \cdots < f^{n-3}(c) < f^{n-1}(c)$$

*and associated digraph is given in the figure, where $J_1 =< c, f(c) >$ and $J_k =< f^{k-2}(c), f^k(c) >$ for $1 < k < n$.*



# Proof of Sharkovski's Theorem

## Proof of Lemma 8

Proof. Decompose FC into two primitive cycles, one of which has odd length. Its length must be 1! Thus FC has the form

$$J_1 \to J_1 \to J_2 \to \cdots \to J_{n-1} \to J_1$$

where $J_i \neq J_1$ for $1 < i < n$. If $J_i = J_k$, $1 < i < k < n$, then we obtain a smaller primitive cycle, and by excluding the loop at $J_1$ if necessary, we can arrange that its length is odd. Hence, $J_1, ..., J_{n-1}$ are all distinct and thus a permutation of $I_1, ..., I_{n-1}$. Similarly, we cannot have $J_i \to J_k$ if $k > i + 1$ or if $k = 1$ and $i \neq 1, n - 1$. Suppose $J_1 = I_k = [a, b]$. Since $\circlearrowright J_1 \to J_2 \Rightarrow J_2$ is adjacent to $J_1$ and either

$$x_k = a,\ x_{k+1} = f(a),\ x_{k-1} = f^2(a)$$

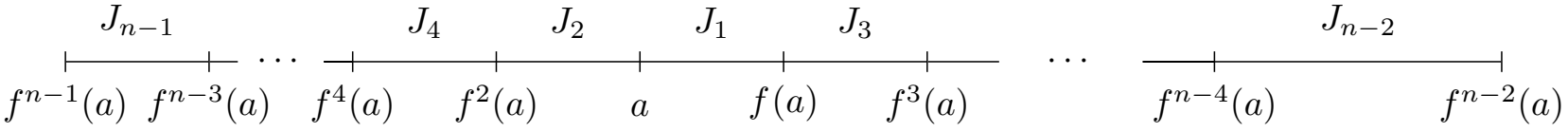
or

$$x_{k+1} = b,\ x_k = f(b),\ x_{k+2} = f^2(b).$$

# Proof of Sharkovski's Theorem

## Proof of Lemma 8

Consider the first case, the argument in the second being similar. If $f^3(a) < f^2(a)$ then $J_2 \to J_1$, which is forbidden. Hence $f^3(a) > f^2(a)$. Since $J_2$ is not directed to $J_k$ for $k > 3$ it follows that $J_3 = [f(a), f^3(a)]$ is adjacent to $J_1$ on the right. If $f^4(a) > f^3(a)$ then $J_3 \to J_1$, which is forbidden. Hence $f^4(a) < f^2(a)$ and, since $J_3$ is not directed to $J_k$ for $k > 4$, $J_4 = [f^4(a), f^2(a)]$ is adjacent to $J_2$ on the left. Proceeding in this way we see that the order of the $J_i$s on the real line is given by




Since the endpoints of $J_{n-1} = [x_1, x_2]$ are mapped into $a$ and $f^{n-2}(a) = x_n$ we have $J_{n-1} \to J_k$ iff $k$ is odd. We found all the arcs in the digraph. $\square$

# Proof of Sharkovski's Theorem

## Lemma 9

*If $f$ has a periodic orbit of odd period $n > 1$, then it has periodic points of arbitrary even order and periodic points of arbitrary odd order $> n$.*

Proof. We may assume $n$ is minimal. Then digraph is a Stefan graph as in Lemma 8. If $m < n$ is even then

$$J_{n-1} \to J_{n-m} \to J_{n-m+1} \to \cdots \to J_{n-1}$$

is a primitive cycle of length $m$. If $m > n$ is even or odd then

$$J_1 \to J_2 \to \cdots \to J_{n-1} \to J_1 \to J_1 \to \cdots \to J_1$$

is a primitive cycle of length $m$. $\square$