

Evolution of Interfaces for the Reaction-Diffusion Equation. II

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FIT Colloquium

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Cauchy problem for the Reaction-Diffusion equation:

$$\mathcal{L}u \equiv u_t - (u^m)_{xx} + bu^\beta = 0, \quad x \in \mathbb{R}, \quad 0 < t < T, \quad (1)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R} \quad (2)$$

where $m > 1$, $b \in \mathbb{R}$, $\beta > 0$, $0 < T \leq +\infty$, $u_0 \geq 0$, $u_0 \in C(\mathbb{R})$.

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Barenblatt's problem: *Does interface expand, shrink or remain stationary? Find the short-time behavior of the interface function $\eta(t)$ and $u(x, t)$ near $x = \eta(t)$.*

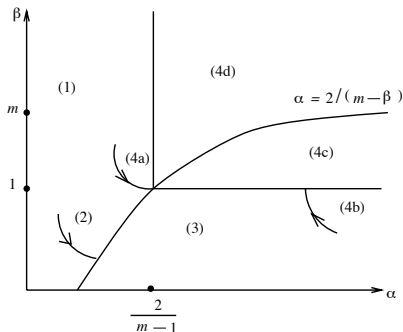
The **Answer** depends on m, β, b, C, α . Is it possible to give a full classification?

Full Classification

U. G. Abdulla, Reaction-diffusion in irregular domains, J. Differential Equations, 164, 2000, 321-354.

U. G. Abdulla and J. R. King, Interface development and local solutions to reaction-diffusion equations, SIAM J. Math. Anal., 32, 2, 2000, 235-260.

U. G. Abdulla, Evolution of interfaces and explicit asymptotics at infinity for the fast diffusion equation with absorption, Nonlinear Analysis, 50, 4, 2002, 541-560.



Lemma 1

Let u solve (1),(2),(4) with $m > 1, 0 < \alpha < \frac{2}{m-1}$. Then

$$u(x, t) = t^{\frac{\alpha}{2+\alpha(1-m)}} f(\xi), \xi = xt^{-\frac{1}{2+\alpha(1-m)}}, f(\xi) = u(\xi, 1)$$

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Let w be a solution of (1),(2),(4) with $C = 1$, and $f_0(\xi) = w(\xi, 1)$ be a corresponding solution of the nonlinear ODE. Then

$$f(\rho) = C^{\frac{2}{2+\alpha(1-m)}} f_0(C^{\frac{m-1}{\alpha(m-1)-2}} \rho), \xi_* = C^{\frac{m-1}{2-\alpha(m-1)}} \xi'_*, \xi'_* = \sup\{\rho : f_0(\rho) > 0\}$$

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If u_0 satisfies (3) then $\eta(t) \sim \xi_* t^{\frac{1}{2+\alpha(1-m)}}$ as $t \downarrow 0$ and $\forall \rho < \xi_*$

$$u(x, t) \sim f(\rho) t^{\frac{\alpha}{2+\alpha(1-m)}} \quad \text{as } t \downarrow 0 \quad \text{along } x = \xi_\rho(t) = \rho t^{\frac{1}{2+\alpha(1-m)}}$$

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Where $b > 0$, $0 < \beta < 1$, $m \geq 1$. Then

$$u(x, t) = t^{\frac{1}{1-\beta}} f_1(\zeta), \quad \zeta = \frac{x}{t^{\frac{m-\beta}{2(1-\beta)}}}$$

$$\begin{cases} (f^m)'' + \frac{m-\beta}{2(1-\beta)} \zeta f' \frac{1}{1-\beta} f - bf^\beta = 0, \zeta \in \mathbb{R} \\ f(\zeta) \sim C(-\zeta)^{\frac{2}{m-\beta}} \quad \text{as } \zeta \downarrow -\infty, \quad f(+\infty) = 0 \end{cases}$$

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If $u_0(x) \sim C(-x)_+^{\frac{2}{m-\beta}}$ as $x \uparrow 0$ and $C > C_* \Rightarrow u(0, t) \sim At^{\frac{1}{1-\beta}}$ as $t \downarrow 0$.

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as

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Reaction-Diffusion in Irregular Domains

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$$\left\{ \begin{array}{l} \text{Dirichlet Problem} \\ u_t - a(u^m)_{xx} + bu^\beta = 0, \quad \text{in } E = \{(x, t) : \phi_1(t) < x < \phi_2(t), 0 < t \leq T\} \\ u(\phi_i(t), t) = \psi_i(t), 0 \leq t \leq T, i = 1, 2 \\ u(x, 0) = u_0(x), \phi_1(0) \leq x \leq \phi_2(0) \\ \phi_i, \psi_i, u_0 \in C, \psi_i, u_0 \geq 0; u_0(\phi_i(0)) = \psi_i(0). \end{array} \right.$$

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- $u \geq 0$ is continuous in \bar{E} , satisfying initial-boundary conditions and $u \in L_\infty(E \cap \{t \leq T_1\})$ for any finite $T_1 \in (0, T]$.
- For any finite t_0, t_1 such that $0 \leq t_0 < t_1 \leq T$ and for any C^∞ functions $\mu_i(t), t_0 \leq t \leq t_1, i = 1, 2$, such that $\phi_1(t) < \mu_1(t) < \mu_2(t) < \phi_2(t)$ for $t \in [t_0; t_1]$ the integral identity

$$\int_{t_0}^{t_1} \int_{\mu_1(t)}^{\mu_2(t)} (u f_t + a u^m f_{xx} - b u^\beta f) dx dt - \int_{\mu_1(t)}^{\mu_2(t)} u f \Big|_{t=t_0}^{t=t_1} dx - \int_{t_0}^{t_1} a u^m f_x \Big|_{x=\mu_1(t)}^{x=\mu_2(t)} dt = 0,$$

holds for arbitrary function $f \in C_{x,t}^{2,1}(\bar{E}_1)$ that equals zero when $x = \mu_i(t), t_0 \leq t \leq t_1$.

Existence and Boundary Regularity

$$\omega_{t_0}^-(\phi; \delta) = \max(\phi(t_0) - \phi(t) : t_0 - \delta \leq t \leq t_0) \quad \text{left modulus of lower semicont.}$$

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Assumption \mathcal{A} : For each $t_0 \in (0, T]$ let there exists a function $F(\delta) > 0, 0 < \delta \ll 1$ with $F(\delta) \rightarrow 0$ as $\delta \downarrow 0$, such that

$$\omega_{t_0}^-(\phi_1; \delta) \leq \delta^{\frac{1}{2}} F(\delta)$$

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Theorem 6

Let Assumption \mathcal{A} hold. Then there exists a solution of the DP.

Assumption \mathcal{M} . Assume for all $0 < \delta \ll 1$

$$\phi_1(t_0) - \phi_1(t) \leq (t - t_0)^\mu \quad \text{for } t_0 \leq t \leq t_0 + \delta,$$

$$\phi_2(t_0) - \phi_2(t) \geq -(t - t_0)^\mu \quad \text{for } t_0 \leq t \leq t_0 + \delta,$$

where $\mu > \frac{1}{2}$ if $0 < m < 1$, and $\mu > \frac{m}{m+1}$ if $m > 1$.

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where $\mu > \frac{1}{2}$ if $0 < m < 1$, and $\mu > \frac{m}{m+1}$ if $m > 1$.

Theorem 7

Let either $m > 0, 0 < \beta < 1, b \geq 0$ or $m > 1, \beta \geq 1$, and b is arbitrary. Let assumption \mathcal{A} hold and there exists a finite number of points $t_i, i = 1, 2, \dots, k$ such that $t_1 = 0 < t_2 < \dots < t_k < t_{k+1} = T$ and for arbitrary compact $[\delta_1, \delta_2] \subset (t_i, t_{i+1}), i = 1, \dots, k$ assumption \mathcal{M} is uniformly satisfied in $[\delta_1, \delta_2]$. Then the solution of the DP is unique.

Definition 8

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- $u \geq 0$ is continuous in \bar{E} , satisfying initial-boundary conditions and $u \in L_\infty(E \cap \{t \leq T_1\})$ for any finite $T_1 \in (0, T]$.
- For any finite t_0, t_1 such that $0 \leq t_0 < t_1 \leq T$ and for any C^∞ functions $\mu_i(t), t_0 \leq t \leq t_1, i = 1, 2$, such that $\phi_1(t) < \mu_1(t) < \mu_2(t) < \phi_2(t)$ for $t \in [t_0; t_1]$ the integral identity

$$\int_{t_0}^{t_1} \int_{\mu_1(t)}^{\mu_2(t)} (u f_t + a u^m f_{xx} - b u^\beta f) dx dt - \int_{\mu_1(t)}^{\mu_2(t)} u f \Big|_{t=t_0}^{t=t_1} dx \\ - \int_{t_0}^{t_1} a u^m f_x \Big|_{x=\mu_1(t)}^{x=\mu_2(t)} dt \leq \quad (\text{or } \geq) \quad 0,$$

holds for arbitrary *nonnegative* function $f \in C_{x,t}^{2,1}(\bar{E}_1)$ that equals zero when $x = \mu_i(t), t_0 \leq t \leq t_1$.

Lemma 9

Let g be a nonnegative and continuous function in \overline{E} belonging to $C_{x,t}^{2,1}$ in E outside a finite number of curves $x = \eta_i(t)$, which divide E into a finite number of subdomains E^j , where $\eta_i \in C[0, T]$;

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$$g_t - a(g^m)_{xx} + bg^\beta \geq 0 \quad (\leq 0)$$

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at the points of E , where $g \in C_{x,t}^{2,1}$. Assume also that the function $(g^m)_x$ is continuous in E and $g \in L_\infty(E \cap (t \leq T_1))$ for any finite $T_1 \in (0, T]$. Then g is a supersolution (subsolution) of the PDE in D .

Theorem 10

Let the conditions of the uniqueness theorem be satisfied. Let u be a solution of the DP, g be a supersolution (or subsolution) of the PDE in E , and

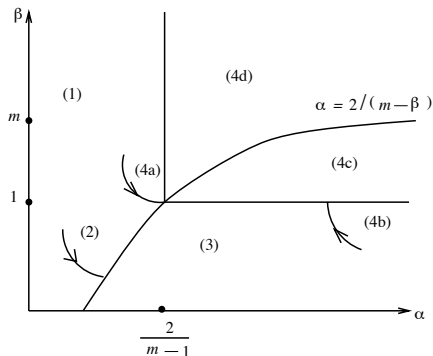
$$u_0(x) \leq \quad (\text{or } \geq) \quad g(x, 0), \quad \text{for } \phi_1(0) \leq x \leq \phi_2(0),$$

$$\psi_i(t) \leq \quad (\text{or } \geq) \quad g(\phi_i(t), t), \quad \text{for } 0 \leq t \leq T.$$

Then

$$u(x, t) \leq \quad (\text{or } \geq) \quad g \quad \text{in } E$$

Main Results



Cases:

- I $b \neq 0$ ($b > 0, \beta > 0$ or $b < 0, \beta \geq 1$) and $m \geq 1$
(Figure above)
- II $b = 0$

Proof of Main Results

Part I, $b \neq 0$ and $m > 1$; 1) $\alpha < 2/(m - \min\{1, \beta\})$

Case I.1) Suppose $\alpha < 2/(m - \min\{1, \beta\})$.

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$$u(x, t) \sim f(\rho)t^{\alpha/(2-\alpha(m-1))}, \quad t \downarrow 0; \quad x = \xi_\rho(t) = \rho t^{1/(2-\alpha(m-1))}$$

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For $\epsilon > 0$, let u_ϵ solve

$$\begin{cases} u_t - (u^m)_{xx} = 0, & x \in \mathbb{R}, \quad 0 < t < T, \\ u_0(x) = (C + \epsilon)(-x)_+^\alpha \end{cases}$$

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Case I.1 is proved

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$$u(x, t) = t^{1/(1-\beta)} f_1(\zeta), \quad \zeta = xt^{-\frac{m-\beta}{2(1-\beta)}}$$

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$$\gamma_0 = 2/(m - \beta) \Rightarrow$$

$$L^0 f_1 = b C_0^\beta (\zeta_0 - \zeta)^{\frac{2\beta}{m-\beta}} \left\{ 1 - \left(\frac{C_0}{C_*} \right)^{m-\beta} + \frac{C_0^{1-\beta}}{b(1-\beta)} \zeta_0 (\zeta_0 - \zeta)^{\frac{2-m-\beta}{m-\beta}} \right\}$$

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From theorem 10,

$$u \leq C_2 t^{\frac{1}{1-\beta}} (\zeta_2 - \zeta)_+^{\frac{2}{m-\beta}}$$

Upper estimation is proved for $C > C_*$.

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Part I; 2) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha = 2/(m - \beta)$, $u_0 = C(-x)_+^\alpha$, $m + \beta \neq 2$, $C > C_*$

By choosing C_0 , ζ_0 , when $m + \beta > 2$ or $1 \leq m < 2 - \beta$,

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g is a subsolution for $x, t > 0$.

As before, from theorem 10,

$$C_1 t^{\frac{1}{1-\beta}} (\zeta_1 - \zeta)_+^\mu \leq u$$

$$\mu = \begin{cases} (m-1)^{-1}, & m + \beta > 2 \\ 2(m-\beta)^{-1}, & 1 \leq m < 2 - \beta \end{cases}$$

Lower estimation is proved when $C > C_*$

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Part I; 2) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha = 2/(m - \beta)$, $u_0 = C(-x)_+^\alpha$, $m + \beta > 2$, $0 < C < C_*$

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Estimate Lg in

$$M = \{(x, t) : -\infty < x < \mu_\gamma(t), t > 0\},$$

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$Lg = bg^\beta S$ where

$$S = \gamma - 2mb^{-1}(m-\beta)^{-2}(2-m-\beta)C^{m-\beta} \left[1 - \frac{b(1-\beta)(1-\gamma)t}{C^{1-\beta}(-x)_+^{\frac{2(1-\beta)}{m-\beta}}} \right]^{\frac{m-1}{1-\beta}} - \\ - 4mb^{-1}(m-\beta)^{-2}(m+\beta-1)C^{m-\beta} \left[1 - \frac{b(1-\beta)(1-\gamma)t}{C^{1-\beta}(-x)_+^{\frac{2(1-\beta)}{m-\beta}}} \right]^{\frac{m+\beta-2}{1-\beta}}$$

Proof of Main Results

Part I; 2) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha = 2/(m - \beta)$, $u_0 = C(-x)_+^\alpha$, $m + \beta > 2$, $0 < C < C_*$

$$S|_{t=0} = \gamma - \left(\frac{C}{C_*}\right)^{m-\beta}, \quad S|_{x=\mu_\gamma(t)} = \gamma$$

Proof of Main Results

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$$\gamma = (C/C_*)^{m-\beta} \text{ (resp. } \gamma = 0) \Rightarrow Lg \geq 0 \text{ (resp. } Lg \leq 0), \text{ in } M$$

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$$\left[C^{1-\beta}(-x)_+^{\frac{2(1-\beta)}{m-\beta}} - b(1-\beta)t \right]_+^{\frac{1}{1-\beta}} \leq u \leq \left[C^{1-\beta}(-x)_+^{\frac{2(1-\beta)}{m-\beta}} - b(1-\beta)(1 - (C/C_*)^{m-\beta})t \right]_+^{\frac{1}{1-\beta}}$$

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Suppose $1 \leq m < 2 - \beta$ and $0 < C < C_*$. Establish the rough estimate

$$\begin{aligned} \left[C^{1-\beta}(-x)_+^{\frac{2(1-\beta)}{m-\beta}} - b(1-\beta)(1 - (C/C_*)^{m-\beta})t \right]_+^{\frac{1}{1-\beta}} &\leq u(x, t) \leq \\ &\leq C(-x)_+^{\frac{2}{m-\beta}} \text{ for } x \in \mathbb{R}, 0 \leq t < +\infty \end{aligned}$$

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To prove the left estimation, take g with $\gamma = (C/C_*)^{m-\beta}$

$$g(x, t) = \left[C^{1-\beta}(-x)_+^{\frac{2(1-\beta)}{m-\beta}} - b(1-\beta)(1 - \gamma)t \right]_+^{\frac{1}{1-\beta}}, \quad x \in \mathbb{R}, t > 0$$

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and derive similarly

$$Lg \leq 0, \text{ in } M; \quad Lg = 0 \text{ for } x > \mu_\gamma(t), t > 0$$

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$$\left[C^{1-\beta}(-x)_+^{\frac{2(1-\beta)}{m-\beta}} - b(1-\beta)(1 - (C/C_*)^{m-\beta})t \right]_+^{\frac{1}{1-\beta}} \leq u(x, t) \leq \\ \leq C(-x)_+^{\frac{2}{m-\beta}} \text{ for } x \in \mathbb{R}, 0 \leq t < +\infty$$

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and derive similarly

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To prove the right-hand side it is enough to observe that

$$Lu_0 = bu_0^\beta(1 - (C/C_*)^{m-\beta}) \geq 0 \text{ for } x \in \mathbb{R}, t \geq 0$$

Proof of Main Results

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To prove an accurate estimation consider

$$g(x, t) = C_0 \left(-\zeta_0 t^{\frac{m-\beta}{2(1-\beta)}} - x \right)_+^{\frac{2}{m-\beta}} \quad \text{in } G_l$$

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$$\text{in } G_l^+ = \left\{ (x, t) : \zeta(t) < x < -\zeta_0 t^{\frac{m-\beta}{2(1-\beta)}}, 0 < t < +\infty \right\}$$

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If $C_0 = C_*$ then $Lg \leq 0$ in G_l^+ ; $Lg = 0$ in $G_l \setminus \bar{G}_l^+$

Proof of Main Results

Part I; 2) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha = 2/(m - \beta)$, $u_0 = C(-x)_+^\alpha$, $1 \leq m < 2 - \beta$, $0 < C < C_*$

Choose ζ_0 , l , and use the rough estimation to show

$$g(\zeta(t), t) = C_*(l_0 - \zeta_3)^{\frac{2}{m-\beta}} t^{\frac{1}{1-\beta}} \leq u(\zeta(t), t), \quad t \geq 0$$

$$g(x, 0) = u(x, 0) = 0, \quad 0 \leq x \leq x_0$$

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Apply comparison thm 10 in $G'_{l_0} = G_{l_0} \cap \{x < x_0\}$ to derive

$$C_* \left(-\zeta_3 t^{\frac{m-\beta}{2(1-\beta)}} - x \right)_+^{\frac{2}{m-\beta}} \leq u$$

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Derive the upper estimation: $S_x \geq 0$ for $\zeta(t) < x < -\zeta_0 t^{\frac{m-\beta}{2(1-\beta)}}$

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$$S \geq S \Big|_{x=\zeta(t)} = 1 - (C_0/C_*)^{m-\beta} - (b(1-\beta))^{-1} C_0^{1-\beta} \zeta_0 (l - \zeta_0)^{\frac{2-m-\beta}{m-\beta}}$$

Proof of Main Results

Part I; 2) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha = 2/(m - \beta)$, $u_0 = C(-x)_+^\alpha$, $1 \leq m < 2 - \beta$, $0 < C < C_*$

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Choose C_0, ζ_0, l to find $S|_{x=\zeta(t)} = 0$

$$Lg \geq 0 \text{ in } G_{l_1}^+, \quad Lg = 0 \text{ in } G_{l_1} \setminus \bar{G}_{l_1}^+$$

$$u(\zeta(t), t) \leq Cl_1^{\frac{2}{m-\beta}} = C_3(l_1 - \zeta_4)^{\frac{2}{m-\beta}} t^{\frac{1}{1-\beta}} = g(\zeta(t), t), \quad t \geq 0$$

Proof of Main Results

Part I; 2) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha = 2/(m - \beta)$, $u_0 = C(-x)_+^\alpha$, $1 \leq m < 2 - \beta$, $0 < C < C_*$

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Apply comparison thm 10 in G'_{l_1} to derive

$$u \leq C_3 \left(-\zeta_4 t^{\frac{m-\beta}{2(1-\beta)}} - x \right)_+^{\frac{2}{m-\beta}}, \quad 0 \leq t < +\infty$$

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$$\zeta_1 t^{\frac{m-\beta}{2(1-\beta)}} \leq \eta(t) \leq \zeta_2 t^{\frac{m-\beta}{2(1-\beta)}}, \quad 0 \leq t < +\infty$$

Proof of Main Results

Part I; 2) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha = 2/(m - \beta)$, $u_0 = C(-x)_+^\alpha$, $1 \leq m < 2 - \beta$, $0 < C < C_*$

Choose C_0, ζ_0, l to find $S|_{x=\zeta(t)} = 0$

$$Lg \geq 0 \text{ in } G_{l_1}^+, \quad Lg = 0 \text{ in } G_{l_1} \setminus \bar{G}_{l_1}^+$$

$$u(\zeta(t), t) \leq Cl_1^{\frac{2}{m-\beta}} = C_3(l_1 - \zeta_4)^{\frac{2}{m-\beta}} t^{\frac{1}{1-\beta}} = g(\zeta(t), t), \quad t \geq 0$$

Apply comparison thm 10 in G'_{l_1} to derive

$$u \leq C_3 \left(-\zeta_4 t^{\frac{m-\beta}{2(1-\beta)}} - x \right)_+^{\frac{2}{m-\beta}}, \quad 0 \leq t < +\infty$$

$$\zeta_1 t^{\frac{m-\beta}{2(1-\beta)}} \leq \eta(t) \leq \zeta_2 t^{\frac{m-\beta}{2(1-\beta)}}, \quad 0 \leq t < +\infty$$

$$\eta(t) = \zeta_* t^{\frac{m-\beta}{2(1-\beta)}}, \quad 0 \leq t < +\infty$$

Proof of Main Results

Part I; 2) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha = 2/(m - \beta)$, $u_0 \sim C(-x)_+^\alpha$, $C \neq C_*$

Suppose $u_0 \sim C(-x)_+^\alpha$, $C \neq C_*$

Proof of Main Results

Part I; 2) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha = 2/(m - \beta)$, $u_0 \sim C(-x)_+^\alpha$, $C \neq C_*$

Suppose $u_0 \sim C(-x)_+^\alpha$, $C \neq C_*$. Similarly to the proof of lemma 1,

$$\eta(t) \sim \zeta_* t^{\frac{m-\beta}{2(1-\beta)}} \text{ as } t \rightarrow 0^+$$

Proof of Main Results

Part I; 2) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha = 2/(m - \beta)$, $u_0 \sim C(-x)_+^\alpha$, $C \neq C_*$

Suppose $u_0 \sim C(-x)_+^\alpha$, $C \neq C_*$. Similarly to the proof of lemma 1,

$$\eta(t) \sim \zeta_* t^{\frac{m-\beta}{2(1-\beta)}} \text{ as } t \rightarrow 0^+$$

$$\rho < \zeta_* \Rightarrow u(x, t) \sim f_1(\rho) t^{1/(1-\beta)} \text{ for } x = \rho t^{\frac{m-\beta}{2(1-\beta)}}, t \rightarrow 0^+$$

Case I.2 is proved.

Proof of Main Results

Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$

Case I.3) Suppose $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$.

Proof of Main Results

Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$

Case I.3) Suppose $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$.

Let $\epsilon > 0$; $u_0 \sim C(-x)_+^\alpha \Rightarrow$

$$(C - \epsilon/2)(-x)_+^\alpha \leq u_0(x) \leq (C + \epsilon/2)(-x)_+^\alpha, \quad x \geq x_\epsilon$$

Proof of Main Results

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$$g_\epsilon(x, t) = \left[(C + \epsilon)^{1-\beta} (-x)_+^{\alpha(1-\beta)} - (1-\beta)(1-\epsilon)t \right]_+^{1/(1-\beta)}$$

Proof of Main Results

Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$

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Estimate Lg in

$$M_1 = \{ (x, t) : x_\epsilon < x < \eta_{l(\epsilon)}(t), \quad 0 < t \leq \delta_1 \}$$

Proof of Main Results

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Estimate Lg in

$$M_1 = \{ (x, t) : x_\epsilon < x < \eta_{l(\epsilon)}(t), \quad 0 < t \leq \delta_1 \}$$

$$\eta_l(t) = -lt^{1/\alpha(1-\beta)}, \quad l(\epsilon) = (C + \epsilon)^{-1/\alpha} [b(1-\beta)(1-\epsilon)]^{1/\alpha(1-\beta)}$$

Proof of Main Results

Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$

Case I.3) Suppose $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$.

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$$Lg_\epsilon = bg_\epsilon^\beta \{\epsilon + S\}$$

Proof of Main Results

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$$S = -b^{-1}m\alpha(C + \epsilon)^{m-\beta}(-x)^{\alpha(m-\beta)-2} \{g|x|^{-\alpha} / (C + \epsilon)\}^{m+\beta-2} S_1$$

Proof of Main Results

Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$

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Estimate Lg in

$$M_1 = \{(x, t) : x_\epsilon < x < \eta_l(\epsilon)(t), \quad 0 < t \leq \delta_1\}$$

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$$S = -b^{-1}m\alpha(C + \epsilon)^{m-\beta}(-x)^{\alpha(m-\beta)-2} \{g|x|^{-\alpha}/(C + \epsilon)\}^{m+\beta-2} S_1$$

$$S_1 = \left\{ (\alpha(1 - \beta) - 1) [g|x|^{-\alpha}/(C + \epsilon)]^{1-\beta} + \alpha(m + \beta - 1) \right\}$$

Proof of Main Results

Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$, $m + \beta \geq 2$

Suppose $m + \beta \geq 2$.

Proof of Main Results

Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$, $m + \beta \geq 2$

Suppose $m + \beta \geq 2$. Then $|S| < \epsilon/2$ in M_1 so

$$Lg_\epsilon > b(\epsilon/2)g_\epsilon^\beta \quad (\text{resp. } Lg_{-\epsilon} < -b(\epsilon/2)g_{-\epsilon}^\beta) \quad \text{in } M_1$$

Proof of Main Results

Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$, $m + \beta \geq 2$

Suppose $m + \beta \geq 2$. Then $|S| < \epsilon/2$ in M_1 so

$$Lg_\epsilon > b(\epsilon/2)g_\epsilon^\beta \quad (\text{resp. } Lg_{-\epsilon} < -b(\epsilon/2)g_{-\epsilon}^\beta) \quad \text{in } M_1$$
$$Lg_{\pm\epsilon} = 0 \quad \text{for } x > \eta_{l(\pm\epsilon)}(t), \quad 0 < t \leq \delta_1$$

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$$g_\epsilon(x, 0) \geq u_0(x) \quad (\text{resp. } g_{-\epsilon}(x, 0) \leq u_0(x)), \quad x \geq x_\epsilon$$

Proof of Main Results

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Choose $\delta \in (0, \delta_1]$ such that

$$g_\epsilon(x_\epsilon, t) \geq u(x_\epsilon, t) \quad (\text{resp. } g_{-\epsilon}(x_\epsilon, t) \leq u(x_\epsilon, t)), \quad 0 \leq t \leq \delta$$

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Proof of Main Results

Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$, $m + \beta \geq 2$

Suppose $m + \beta \geq 2$. Then $|S| < \epsilon/2$ in M_1 so

$$Lg_\epsilon > b(\epsilon/2)g_\epsilon^\beta \quad (\text{resp. } Lg_{-\epsilon} < -b(\epsilon/2)g_{-\epsilon}^\beta) \quad \text{in } M_1$$

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$$\eta_{l(-\epsilon)}(t) \leq \eta(t) \leq \eta_{l(\epsilon)}(t), \quad 0 \leq t \leq \delta$$

Proof of Main Results

Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$, $m + \beta \geq 2$

Suppose $m + \beta \geq 2$. Then $|S| < \epsilon/2$ in M_1 so

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$$\eta_{l(-\epsilon)}(t) \leq \eta(t) \leq \eta_{l(\epsilon)}(t), \quad 0 \leq t \leq \delta$$

$$\eta(t) \sim -l_* t^{1/\alpha(1-\beta)}, \quad t \rightarrow 0^+, \quad l_* = C^{-1/\alpha} (b(1-\beta))^{1/(\alpha(1-\beta))}$$

Proof of Main Results

Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$, $m + \beta \geq 2$

Suppose $m + \beta \geq 2$. Then $|S| < \epsilon/2$ in M_1 so

$$Lg_\epsilon > b(\epsilon/2)g_\epsilon^\beta \quad (\text{resp. } Lg_{-\epsilon} < -b(\epsilon/2)g_{-\epsilon}^\beta) \quad \text{in } M_1$$

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$$u(x, t) \sim \left[C^{1-\beta} (-x)_+^{\alpha(1-\beta)} - b(1-\beta)t \right]^{1/(1-\beta)} \quad \text{as } t \rightarrow 0^+$$

Proof of Main Results

Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$, $1 \leq m < 2 - \beta$

Suppose $1 \leq m < 2 - \beta$.

Proof of Main Results

Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$, $1 \leq m < 2 - \beta$

Suppose $1 \leq m < 2 - \beta$. Lower est. is similar. Prove an upper estimation:

$$g(x, t) = C_6 \left(-\zeta_5 t^{\frac{1}{\alpha(1-\beta)}} - x \right)_+^\alpha \text{ in } G_{l,\delta},$$
$$G_{l,\delta} = \{(x, t) : \eta_l(t) < x < +\infty, 0 < t < \delta\}$$

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Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$, $1 \leq m < 2 - \beta$

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For $l > l_*$, $\epsilon > 0$, $\exists \delta = \delta(l, \epsilon)$ s.t.

$$u(\eta_l(t), t) \leq \left[C^{1-\beta} l^{\alpha(1-\beta)} - b(1-\beta)(1-\epsilon) \right]^{\frac{1}{1-\beta}} t^{\frac{1}{1-\beta}}, \quad 0 \leq t \leq \delta.$$

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$$Lg \text{ in } G_{l,\delta}^+ = \left\{ (x, t) : \eta_l(t) < x < -\zeta_5 t^{\frac{1}{\alpha(1-\beta)}}, 0 < t < \delta \right\} :$$

Proof of Main Results

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Suppose $1 \leq m < 2 - \beta$. Lower est. is similar. Prove an upper estimation:

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$$Lg \text{ in } G_{l, \delta}^+ = \left\{ (x, t) : \eta_l(t) < x < -\zeta_5 t^{\frac{1}{\alpha(1-\beta)}}, 0 < t < \delta \right\} :$$

$$Lg = bg^\beta S, \quad S = 1 - (b(1-\beta))^{-1} C_6^{1/\alpha} \zeta_5 \{gt^{1/(\beta-1)}\}^{1-\beta-1/\alpha} -$$
$$-b^{-1} \alpha m (\alpha m - 1) C_6^{2/\alpha} g^{m-\beta-2/\alpha}$$

Proof of Main Results

Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$, $1 \leq m < 2 - \beta$

$$S_x \geq 0 \text{ in } G_{l,\delta}^+$$

Proof of Main Results

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$$S_x \geq 0 \text{ in } G_{l,\delta}^+$$

$$S \geq S \Big|_{x=\eta_l(t)} \geq \epsilon^{-b^{-1}} C_6^{m-\beta} \alpha m (\alpha m - 1) \left\{ (l - \zeta_5) t^{1/\alpha(1-\beta)} \right\}^{\alpha(m-\beta)-2} \text{ in } G_{l,\delta}^+$$

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Choose δ s.t. $Lg \geq b(\epsilon/2)g^\beta$ in $G_{l,\delta}^+$

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Choose δ s.t. $Lg \geq b(\epsilon/2)g^\beta$ in $G_{l,\delta}^+$

$$Lg = 0 \text{ in } G'_{l,\delta} \setminus \bar{G}_{l,\delta}^+$$

Proof of Main Results

Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$, $1 \leq m < 2 - \beta$

$$S_x \geq 0 \text{ in } G_{l,\delta}^+$$

$$S \geq S \Big|_{x=\eta_l(t)} \geq \epsilon - b^{-1} C_6^{m-\beta} \alpha m (\alpha m - 1) \left\{ (l - \zeta_5) t^{1/\alpha(1-\beta)} \right\}^{\alpha(m-\beta)-2} \text{ in } G_{l,\delta}^+$$

Choose δ s.t. $Lg \geq b(\epsilon/2)g^\beta$ in $G_{l,\delta}^+$

$$Lg = 0 \text{ in } G'_{l,\delta} \setminus \bar{G}_{l,\delta}^+$$

$$\begin{aligned} u(\eta_l(t), t) &\leq \left[C^{1-\beta} l^{\alpha(1-\beta)} - b(1-\beta)(1-\epsilon) \right]^{\frac{1}{1-\beta}} t^{\frac{1}{1-\beta}} \\ &= C_6 (l - \zeta_5)^{\alpha} t^{\frac{1}{\alpha(1-\beta)}} = g(\eta_l(t), t), \quad 0 \leq t \leq \delta \end{aligned}$$

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$$S \geq S \Big|_{x=\eta_l(t)} \geq \epsilon - b^{-1} C_6^{m-\beta} \alpha m (\alpha m - 1) \left\{ (l - \zeta_5) t^{1/\alpha(1-\beta)} \right\}^{\alpha(m-\beta)-2} \text{ in } G_{l,\delta}^+$$

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$$Lg = 0 \text{ in } G'_{l,\delta} \setminus \bar{G}_{l,\delta}^+$$

$$\begin{aligned} u(\eta_l(t), t) &\leq \left[C^{1-\beta} l^{\alpha(1-\beta)} - b(1-\beta)(1-\epsilon) \right]^{\frac{1}{1-\beta}} t^{\frac{1}{1-\beta}} \\ &= C_6 (l - \zeta_5)^{\alpha} t^{\frac{1}{\alpha(1-\beta)}} = g(\eta_l(t), t), \quad 0 \leq t \leq \delta \end{aligned}$$

$$u(x_0, t) = g(x_0, t) = 0, \quad 0 \leq t \leq \delta$$

$$u(x, 0) = g(x, 0) = 0, \quad 0 \leq x \leq x_0$$

Proof of Main Results

Part I; 3) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m - \beta)$, $1 \leq m < 2 - \beta$

Comparison theorem 10 \Rightarrow

$$u(x, t) \leq C_6 \left(-\zeta_5 t^{\frac{1}{\alpha(1-\beta)}} - x \right)_+^\alpha \text{ in } \bar{G}_{l,\delta}$$

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$$-lt^{1/\alpha(1-\beta)} \leq \eta(t) \leq -\zeta_5 t^{1/\alpha(1-\beta)}, \quad 0 \leq t \leq \delta$$

where $\zeta_5 = \left(\frac{l_*}{l} \right)^{\alpha(1-\beta)} (1 - \epsilon)l$.

$$\eta(t) \sim -l_* t^{1/\alpha(1-\beta)}, \quad t \rightarrow 0^+, \quad l_* = C^{-1/\alpha} (b(1 - \beta))^{1/(\alpha(1-\beta))}$$

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$$\eta(t) \sim -l_* t^{1/\alpha(1-\beta)}, \quad t \rightarrow 0^+, \quad l_* = C^{-1/\alpha} (b(1 - \beta))^{1/(\alpha(1-\beta))}$$

$$u(x, t) \sim \left[C^{1-\beta} (-x)_+^{\alpha(1-\beta)} - b(1 - \beta)t \right]^{1/(1-\beta)} \text{ as } t \rightarrow 0^+$$

along $x = \eta_l(t) = -lt^{1/\alpha(1-\beta)}$, $l > l_*$. Case I.3 is proved.

Proof of Main Results

Part I; 4b) $\beta = 1, \alpha > 2/(m - 1)$

Case 4a) is immediate.

Case 4b) Suppose $\beta = 1, \alpha > 2/(m - 1)$.

$$\forall 0 < \epsilon \ll 1 \Rightarrow (C - \epsilon/2)(-x)_+^\alpha \leq u_0(x) \leq (C + \epsilon/2)(-x)_+^\alpha, \quad x \geq x_\epsilon$$

$$g(x, t) = (C - \epsilon)(-x)_+^\alpha \exp(-bt)$$

$$Lg \leq 0 \text{ for } x_\epsilon < x < 0, t > 0; \quad Lg = 0 \text{ for } x, t > 0$$

$$\text{Choose } \delta \text{ s.t. } g(x_\epsilon, t) \leq u(x_\epsilon, t) \text{ for } 0 \leq t \leq \delta_\epsilon$$

$$(C - \epsilon)(-x)_+^\alpha \exp(-bt) \leq u(x, t) \text{ for } x \geq x_\epsilon, 0 \leq t \leq \delta_\epsilon$$

Lower Estimate is Proved

Proof of Main Results

Part I; 4b) $\beta = 1, \alpha > 2/(m - 1)$

$$g(x, t) = (C + \epsilon)(-x)_+^\alpha \exp(-bt) \cdot$$

$$\cdot \left[1 - \epsilon(b(m - 1))^{-1} (1 - \exp(-b(m - 1)t)) \right]^{1/(1-m)}$$

$|x_\epsilon| \ll 1 \Rightarrow Lg \geq 0$ for $x_\epsilon < x < 0, t > 0$; $Lg = 0$ for $x, t > 0$

$$u(x, t) \leq (C + \epsilon)(-x)_+^\alpha \exp(-bt) \cdot$$

$$\cdot \left[1 - \epsilon(b(m - 1))^{-1} (1 - \exp(-b(m - 1)t)) \right]^{1/(1-m)}$$

Case I.4b is proved. (4c and 4d are similar.)

Proof of Main Results

Part II; 1) $b = 0$, $m > 1$, $0 < \alpha < 2/(m - 1)$

Part II ($b = 0$), Case 1). Suppose $m > 1$, $0 < \alpha < 2/(m - 1)$.

$$u_0(x) = C(-x)_+^\alpha \Rightarrow u(x, t) = t^{\frac{\alpha}{2+\alpha(1-m)}} f(\xi), \quad \xi = xt^{-\frac{1}{2+\alpha(1-m)}}$$

$$\eta(t) = \xi_* t^{\frac{1}{2+\alpha(1-m)}} \text{ for } 0 \leq t < +\infty$$

Consider

$$g(x, t) = t^{\alpha/(2-\alpha(m-1))} f(\xi)$$

$$Lg = t^{(m\alpha-2)/(2-\alpha(m-1))} L_t f$$

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$$f(\xi) = C_0(\xi_0 - \xi)_+^{1/(m-1)}$$

Proof of Main Results

Part II; 1) $b = 0$, $m > 1$, $0 < \alpha < 2/(m - 1)$, $u_0(x) = C(-x)_+^\alpha$

$$L_t f = (2 - \alpha(m - 1))^{-1} (m - 1)^{-1} C_0 (\xi_0 - \xi)^{\frac{2-m}{m-1}} R(\xi) \\ \text{for } 0 \leq \xi < \xi_0, t > 0$$

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$$R(\xi) = \alpha(m-1)\xi_0 + (1 - \alpha(m-1))\xi - (m-1)^{-1}m(2 - \alpha(m-1))C_0^{m-1}$$

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$$R(\xi) \geq \nu_\alpha \xi_4 - (m - 1)m(2 - \alpha(m - 1))C_5^{m-1} = 0$$

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$$u(x, t) \leq C_5 t^{\frac{\alpha}{2+\alpha(1-m)}} (\xi_4 - \xi)_+^{\frac{1}{m-1}}$$

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$$C_4 t^{\frac{\alpha}{2+\alpha(1-m)}} (\xi_3 - \xi)_+^{\frac{1}{m-1}} \leq u(x, t)$$

Proof of Main Results

Part II

Remaining results, and case $u_0 \sim C(-x)_+^\alpha$ are established using lemma 1.
Part II ($b = 0$), Case 2) Suppose $m > 1$, $\alpha = 2/(m - 1)$

Proof of Main Results

Part II

Remaining results, and case $u_0 \sim C(-x)_+^\alpha$ are established using lemma 1.

Part II ($b = 0$), Case 2) Suppose $m > 1$, $\alpha = 2/(m - 1)$

$$u_0(x) = C(-x)_+^\alpha \Rightarrow$$

$$u_C(x, t) = C(-x)_+^\alpha [1 - (C/\bar{C})^{m-1}(m-1)t]^{1/(1-m)}, \quad x \in \mathbb{R}, \quad 0 \leq t < T$$

$$u_0(x) \sim C(-x)_+^\alpha \Rightarrow$$

$$u_{C-\epsilon} \leq u \leq u_{C+\epsilon}$$

Part II ($b = 0$), Case 3) Suppose $m > 1$, $\alpha > 2/(m - 1)$

For all $\epsilon > 0 \exists x_\epsilon < 0$, $\delta_\epsilon > 0$ s.t.

$$(C - \epsilon)(-x)_+^\alpha \leq u \leq (C + \epsilon)(-x)_+^\alpha (1 - \epsilon t)^{1/(1-m)}, \quad x_\epsilon \leq x, \quad 0 \leq t \leq \delta_\epsilon$$

□

Open Problems

Interfaces for the Reaction-Diffusion-Convection Equations

Cauchy problem for the Reaction-Diffusion-Convection equation:

$$u_t - (u^m)_{xx} + bu^\beta + c(u^\gamma)_x = 0, \quad x \in \mathbb{R}, \quad 0 < t < T, \quad (13)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R} \quad (14)$$

where $m > 0$, $b, c \in \mathbb{R}$, $\beta, \gamma > 0$, $0 < T \leq +\infty$, $u_0 \geq 0$, $u_0 \in C(\mathbb{R})$.

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$$\eta(t) = \sup\{x : u(x, t) > 0\}, \quad \eta(0) = 0$$

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$$u_0(x) \sim C(-x)_+^\alpha, \quad \text{as } x \rightarrow 0-, \quad \text{for some } C > 0, \alpha > 0. \quad (15)$$

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$$u_0(x) = C(-x)_+^\alpha, \quad x \in \mathbb{R} \quad (16)$$

Open Problems

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Does interface expand, shrink or remain stationary? Find the short-time behavior of the interface function $\eta(t)$ and $u(x, t)$ near $x = \eta(t)$.

The **Answer** depends on $m, \beta, \gamma, b, c, C, \alpha$. Is it possible to give a full classification?

Open Problems

Interfaces for the Anisotropic Reaction-Diffusion Equations

Nonhomogeneous Reaction-Diffusion-Convection equation:

$$u_t - (a(x,t)u^m)_{xx} + b(x,t)u^\beta + c(x,t)(u^\gamma)_x = 0, \quad x \in \mathbb{R}, \quad 0 < t < T, \quad (17)$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R} \quad (18)$$

where $m, \beta, \gamma > 0$, $a, b, c \in C(\mathbb{R})$.

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where $m, \beta, \gamma > 0$, $a, b, c \in C(\mathbb{R})$.

Reaction-Diffusion-Convection equations with non-standard growth condition:

$$u_t - (u^{m(x)})_{xx} + bu^{\beta(x)} + c(u^{\gamma(x)})_x = 0, \quad x \in \mathbb{R}, \quad 0 < t < T, \quad (19)$$

where $m, \beta, \gamma \in C(\mathbb{R})$.

Open Problems

Interfaces for the Anisotropic Reaction-Diffusion Equations

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$$\eta(t) = \sup\{x : u(x,t) > 0\}, \quad \eta(0) = 0$$

Open Problems

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$$u_0(x) \sim C(-x)_+^\alpha, \quad \text{as } x \rightarrow 0-, \quad \text{for some } C > 0, \alpha > 0. \quad (20)$$

Open Problems

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Find the short-time behavior of the interface function $\eta(t)$ and $u(x,t)$ near $x = \eta(t)$.

Open Problems

Interfaces for the Multidimensional Reaction-Diffusion Equations

Cauchy problem for the Reaction-Diffusion equation:

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$$u_t - \Delta(u^m) + bu^\beta = 0, \quad x \in \mathbb{R}^n, \quad 0 < t < T, \quad (21)$$

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Let $\text{supp}(u_0)$ is compact, $\partial \text{supp}(u_0)$ possibly has a corner or cusp near 0.

Cauchy problem for the Reaction-Diffusion equation:

$$u_t - \Delta(u^m) + bu^\beta = 0, \quad x \in \mathbb{R}^n, \quad 0 < t < T, \quad (21)$$

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Let $\text{supp}(u_0)$ is compact, $\partial \text{supp}(u_0)$ possibly has a corner or cusp near 0.

Given asymptotic behaviour of u_0 near $\partial \text{supp}(u_0)$ find the evolution of the interfaces with possible singularities and $u(x, t)$ near the interface.

Open Problems

Interfaces for the Reaction-Diffusion Systems

Cauchy problem for the Reaction-Diffusion system:

Cauchy problem for the Reaction-Diffusion system:

$$u_t = \Delta(u^{m_1}) + bu^{\beta_1}v^{\gamma_1}, \quad x \in \mathbb{R}^n, \quad 0 < t < T, \quad (23)$$

$$v_t = \Delta(v^{m_2}) - cu^{\beta_2}v^{\gamma_2}, \quad x \in \mathbb{R}^n, \quad 0 < t < T, \quad (24)$$

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Find the short-time behavior of the interface functions $\eta(t)$, $\zeta(t)$ and $u(x, t)$, $v(x, t)$ near $x = \eta(t)$, $x = \zeta(t)$.