# Evolution of Interfaces for the Reaction-Diffusion Equation. II

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# Introduction

Cauchy problem for the Reaction-Diffusion equation:

$$\mathcal{L}u \equiv u_t - (u^m)_{xx} + bu^\beta = 0, \quad x \in \mathbb{R},\ 0 < t < T, \qquad (1)$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R} \qquad (2)$$

where $m > 1$, $b \in \mathbb{R}$, $\beta > 0$, $0 < T \le +\infty$, $u_0 \ge 0$, $u_0 \in C(\mathbb{R})$.

$$\eta(t) = \sup\{x : u(x,t) > 0\}, \quad \eta(0) = 0$$

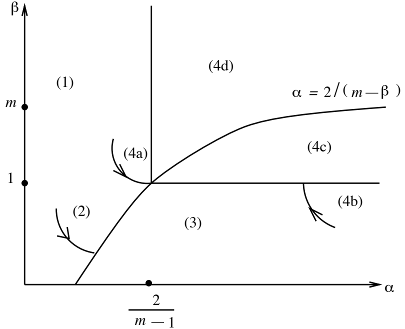
$$u_0(x) \sim C(-x)_+^\alpha, \quad \text{as } x \to 0-, \quad \text{for some } C > 0, \alpha > 0. \qquad (3)$$

$$u_0(x) = C(-x)_+^\alpha, \quad x \in \mathbb{R} \qquad (4)$$

**Barenblatt's problem:** *Does interface expand, shrink or remain stationary? Find the short-time behavior of the interface function $\eta(t)$ and $u(x,t)$ near $x = \eta(t)$.*
The **Answer** depends on $m, \beta, b, C, \alpha$. Is it possible to give a full classification?

# Full Classification

U. G. Abdulla, Reaction-diffusion in irregular domains, J. Differential Equations, 164, 2000, 321-354.
U. G. Abdulla and J. R. King, Interface development and local solutions to reaction-diffusion equations, SIAM J. Math. Anal., 32, 2, 2000, 235-260.
U. G. Abdulla, Evolution of interfaces and explicit asymptotics at infinity for the fast diffusion equation with absorption, Nonlinear Analysis, 50, 4, 2002, 541-560.



## Lemma 1

*Let $u$ solve (1),(2),(4) with $m > 1, 0 < \alpha < \frac{2}{m-1}$. Then*

$$u(x,t) = t^{\frac{\alpha}{2+\alpha(1-m)}} f(\xi), \xi = xt^{-\frac{1}{2+\alpha(1-m)}},\ f(\xi) = u(\xi,1)$$

$$\frac{d^2 f^m}{d\xi^2} + \xi\frac{df}{d\xi} - \frac{\alpha}{2+\alpha(1-m)} f = 0, \xi \in \mathbb{R}; f \sim C(-\xi)^\alpha \textit{as}\, \xi \downarrow -\infty, f(+\infty) = 0,$$

$$\exists \xi_* \quad \textit{such that}\ f(\xi) > 0, -\infty < \xi < \xi_*; f(\xi) \equiv 0, \xi \geq \xi_*,\ \ \eta(t) = \xi_* t^{\frac{1}{2+\alpha(1-m)}}$$

*Let $w$ be a solution of (1),(2),(4) with $C = 1$, and $f_0(\xi) = w(\xi,1)$ be a corresponding solution of the nonlinear ODE. Then*

$$f(\rho) = C^{\frac{2}{2+\alpha(1-m))}} f_0(C^{\frac{m-1}{\alpha(m-1)-2}} \rho), \xi_* = C^{\frac{m-1}{2-\alpha(m-1)}} \xi_*', \xi_*' = \sup\{\rho : f_0(\rho) > 0\}$$

*If $u_0$ satisfies (3) then $\eta(t) \sim \xi_* t^{\frac{1}{2+\alpha(1-m)}}$ as $t \downarrow 0$ and $\forall \rho < \xi_*$*

$$u(x,t) \sim f(\rho) t^{\frac{\alpha}{2+\alpha(1-m)}} \quad \textit{as}\ t \downarrow 0 \quad \textit{along}\ x = \xi_\rho(t) = \rho t^{\frac{1}{2+\alpha(1-m)}}$$

## Lemma 2

*Let* $u$ *solve*

$$u_t = (u^m)_{xx} - bu^\beta,\ x \in \mathbb{R},\ t > 0 \tag{5}$$

$$u(x,0) = u_0(x),\ x \in \mathbb{R} \tag{6}$$

$$u_0 \sim C(-x)_+^\alpha,\ \text{as } x \uparrow 0 \tag{7}$$

a) $b > 0$, $0 < \beta < 1 < m$, $0 < \alpha < 2/(m - \beta)$

b) $b \neq 0$, $\beta \geq 1$, $m > 1$, $0 < \alpha < 2/(m - 1)$

*Then*

$$u(x,t) \sim f(\rho) t^{\frac{\alpha}{2-\alpha(m-1)}}, \quad x = \xi_\rho(t) = \rho t^{\frac{1}{2-\alpha(m-1)}},\ \rho < \xi_*$$

## Lemma 3

*Let $u$ solve*

$$u_t = (u^m)_{xx} - bu^\beta,\ x \in \mathbb{R},\ t > 0 \tag{8}$$

$$u(x,0) = C(-x)_+^\alpha,\ x \in \mathbb{R},\ \alpha = 2/(m-\beta) \tag{9}$$

*Where $b > 0$, $0 < \beta < 1$, $m \geq 1$. Then*

$$u(x,t) = t^{\frac{1}{1-\beta}} f_1(\zeta),\ \zeta = \frac{x}{t^{\frac{m-\beta}{2(1-\beta)}}}$$

$$\begin{cases} (f^m)'' + \frac{m-\beta}{2(1-\beta)}\zeta f' \frac{1}{1-\beta} f - bf^\beta = 0, \zeta \in \mathbb{R} \\ f(\zeta) \sim C(-\zeta)^{\frac{2}{m-\beta}} \quad \text{as } \zeta \downarrow -\infty,\ f(+\infty) = 0 \end{cases}$$

$\exists\zeta_*$ st. $f \equiv 0$ if $\zeta \geq \zeta_*$; $f(\zeta) > 0$, if $\zeta < \zeta_*$.

- If
$$C = C_* \equiv \left[\frac{b(m-\beta)^2}{2m(m+\beta)}\right]^{\frac{1}{m-\beta}} \text{ then } \zeta_* = 0$$
$f(\zeta) = C_*(-\zeta)_*^{\frac{2}{m-\beta}} \Rightarrow u(x,t) = C_*(-x)^{\frac{2}{m-\beta}}$ is a stationary solution.
- If $C \lesseqgtr C_*$ then $\zeta_* \lesseqgtr 0$; [Alvarez & Diaz,92; Herrero & Vazquez, 88]
- If $C > C_* \Rightarrow \zeta_* > 0 \Rightarrow f(0) = A$ where $A(m,\beta,C,b) > 0$ $\Rightarrow u(0,t) = At^{\frac{1}{1-\beta}}$

If $u_0(x) \sim C(-x)_+^{\frac{2}{m-\beta}}$ as $x \uparrow 0$ and $C > C_* \Rightarrow u(0,t) \sim At^{\frac{1}{1-\beta}}$ as $t \downarrow 0$.

## Lemma 4

*Let $u$ solve*

$$u_t = (u^m)_{xx} - bu^\beta,\ x \in \mathbb{R},\ t > 0 \tag{10}$$

$$u(x,0) = u_0(x),\ x \in \mathbb{R} \tag{11}$$

$$u_0 \sim C(-x)_+^\alpha,\ \text{as } x \uparrow 0 \tag{12}$$

*If* $b > 0$, $0 < \beta < 1$, $\alpha > \frac{2}{m-\beta}$, *then* $\forall l > l_* = C^{-1/\alpha}\big(b(1-\beta)\big)^{\frac{1}{1-\beta}}$

$$u(x,t) \sim \Big[C^{1-\beta}(-x)_+^{\alpha(1-\beta)} - b(1-\beta)t\Big]^{\frac{1}{1-\beta}}$$

*as*

$$t \downarrow 0, \quad x = \eta_l(t) = -lt^{\frac{1}{\alpha(1-\beta)}}$$

$$
\begin{cases}
\quad \text{Cauchy Problem} \\
u_t - a(u^m)_{xx} + bu^\beta = 0, \quad \text{in } S = \{(x,t) : x \in \mathbb{R}, 0 < t \le T\} \\
u(x,0) = u_0(x), x \in \mathbb{R} \\
u_0 \in C(\mathbb{R}),\ u_0 \ge 0,\ \sup u_0 < +\infty
\end{cases}
$$

$$
\begin{cases}
\quad \text{Cauchy-Dirichlet Problem} \\
u_t - a(u^m)_{xx} + bu^\beta = 0, \quad \text{in } D = \{(x,t) : s(t) < x < +\infty, 0 < t \le T\} \\
u(s(t),t) = \psi(t), 0 \le t \le T \\
u(x,0) = u_0(x), s(0) \le x < +\infty \\
s, \psi, u_0 \in C, \psi, u_0 \ge 0; \sup u_0 < +\infty, u_0(s(0)) = \psi(0).
\end{cases}
$$

$$
\begin{cases}
\quad \text{Dirichlet Problem} \\
u_t - a(u^m)_{xx} + bu^\beta = 0, \quad \text{in } E = \{(x,t) : \phi_1(t) < x < \phi_2(t), 0 < t \le T\} \\
u(s_i(t),t) = \psi_i(t), 0 \le t \le T,\ i = 1,2 \\
u(x,0) = u_0(x), \phi_1(0) \le x \le \phi_2(0) \\
\phi_i, \psi_i, u_0 \in C, \psi_i, u_0 \ge 0; u_0(\phi_i(0)) = \psi_i(0).
\end{cases}
$$

## Definition 5

We shall say $u(x,t)$ is a solution of the DP if

a) $u \geq 0$ is continuous in $\bar{E}$, satisfying initial-boundary conditions and $u \in L_\infty(E \cap \{t \leq T_1\})$ for any finite $T_1 \in (0,T]$.

b) For any finite $t_0, t_1$ such that $0 \leq t_0 < t_1 \leq T$ and for any $C^\infty$ functions $\mu_i(t), t_0 \leq t \leq t_1, i = 1,2$, such that $\phi_1(t) < \mu_1(t) < \mu_2(t) < \phi_2(t)$ for $t \in [t_0;t_1]$ the integral identity

$$\int_{t_0}^{t_1}\int_{\mu_1(t)}^{\mu_2(t)} (uf_t + au^m f_{xx} - bu^\beta f)dxdt - \int_{\mu_1(t)}^{\mu_2(t)} uf\Big|_{t=t_0}^{t=t_1} dx$$

$$-\int_{t_0}^{t_1} au^m f_x\Big|_{x=\mu_1(t)}^{x=\mu_2(t)} dt = 0,$$

holds for arbitrary function $f \in C^{2,1}_{x,t}(\overline{E_1})$ that equals zero when $x = \mu_i(t), t_0 \leq t \leq t_1$.

# Existence and Boundary Regularity

$\omega^{-}_{t_0}(\phi;\delta) = \max(\phi(t_0) - \phi(t) : t_0 - \delta \le t \le t_0)$ left modulus of lower semicont.

$\omega^{+}_{t_0}(\phi;\delta) = \min(\phi(t_0) - \phi(t) : t_0 - \delta \le t \le t_0)$ left modulus of upper semicont.

**Assumption** $\mathcal{A}$: For each $t_0 \in (0, T]$ let there exists a function $F(\delta) > 0, 0 < \delta << 1$ with $F(\delta) \to 0$ as $\delta \downarrow 0$, such that

$$\omega^{-}_{t_0}(\phi_1;\delta) \le \delta^{\frac{1}{2}} F(\delta)$$

$$\omega^{+}_{t_0}(\phi_2;\delta) \ge -\delta^{\frac{1}{2}} F(\delta)$$

Theorem 6

*Let Assumption $\mathcal{A}$ hold. Then there exists a solution of the DP.*

# Uniqueness

Assumption $\mathcal{M}$. Assume for all $0 < \delta << 1$

$$\phi_1(t_0) - \phi_1(t) \le (t - t_0)^\mu \quad \text{for } t_0 \le t \le t_0 + \delta,$$

$$\phi_2(t_0) - \phi_2(t) \ge -(t - t_0)^\mu \quad \text{for } t_0 \le t \le t_0 + \delta,$$

where $\mu > \frac{1}{2}$ if $0 < m < 1$, and $\mu > \frac{m}{m+1}$ if $m > 1$.

## Theorem 7

*Let either $m > 0, 0 < \beta < 1, b \ge 0$ or $m > 1, \beta \ge 1$, and $b$ is arbitrary. Let assumption $\mathcal{A}$ hold and there exists a finite number of points $t_i, i = 1, 2, ..., k$ such that $t_1 = 0 < t_2 < ... < t_k < t_{k+1} = T$ and for arbitrary compact $[\delta_1, \delta_2] \subset (t_i, t_{i+1}), i = 1, ..., k$ assumption $\mathcal{M}$ is uniformly satisfied in $[\delta_1, \delta_2]$. Then the solution of the DP is unique.*

# Super- and Subsolutions

## Definition 8

We shall say $u(x,t)$ is a *supersolution* (or *subsolution*) of the DP if

a) $u \geq 0$ is continuous in $\bar{E}$, satisfying initial-boundary conditions and $u \in L_\infty(E \cap \{t \leq T_1\})$ for any finite $T_1 \in (0,T]$.

b) For any finite $t_0, t_1$ such that $0 \leq t_0 < t_1 \leq T$ and for any $C^\infty$ functions $\mu_i(t), t_0 \leq t \leq t_1, i = 1,2$, such that $\phi_1(t) < \mu_1(t) < \mu_2(t) < \phi_2(t)$ for $t \in [t_0; t_1]$ the integral identity

$$\int_{t_0}^{t_1}\int_{\mu_1(t)}^{\mu_2(t)} (uf_t + au^m f_{xx} - bu^\beta f)dxdt - \int_{\mu_1(t)}^{\mu_2(t)} uf\Big|_{t=t_0}^{t=t_1} dx$$

$$-\int_{t_0}^{t_1} au^m f_x \Big|_{x=\mu_1(t)}^{x=\mu_2(t)} dt \leq \quad (\text{or } \geq)\ 0,$$

holds for arbitrary *nonnegative* function $f \in C^{2,1}_{x,t}(\overline{E_1})$ that equals zero when $x = \mu_i(t), t_0 \leq t \leq t_1$.

# Lemma on Super- and Subsolutions

## Lemma 9

*Let $g$ be a nonnegative and continuous function in $\overline{E}$ belonging to $C^{2.1}_{x,t}$ in $E$ outside a finite number of curves $x = \eta_i(t)$, which divide $E$ into a finite number of subdomains $E^j$, where $\eta_i \in C[0,T]$; for arbitrary $\delta > 0$ and finite $\Delta \in (\delta, T]$ the function $\eta_i$ is absolutely continuous in $[\delta, \Delta]$. Let $g$ satisfy the inequality*

$$g_t - a(g^m)_{xx} + bg^\beta \geq 0 \ (\leq 0)$$

*at the points of $E$, where $g \in C^{2,1}_{x,t}$. Assume also that the function $(g^m)_x$ is continuous in $E$ and $g \in L_\infty(E \cap (t \leq T_1))$ for any finite $T_1 \in (0, T]$. Then $g$ is a supersolution (subsolution) of the PDE in $D$.*

# Comparison Theorem

### Theorem 10

*Let the conditions of the uniqueness theorem be satisfied. Let $u$ be a solution of the DP, $g$ be a supersolution (or subsolution) of the PDE in $E$, and*

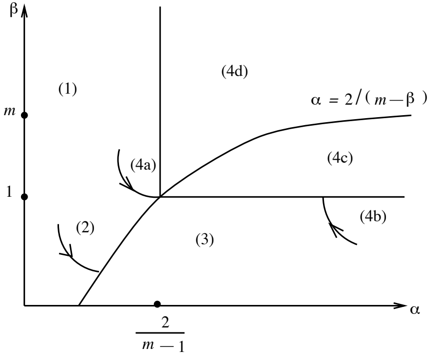
$$u_0(x) \leq \quad \text{(or } \geq \text{)}\ g(x,0), \quad \text{for } \phi_1(0) \leq x \leq \phi_2(0),$$

$$\psi_i(t) \leq \quad \text{(or } \geq \text{)}\ g(\phi_i(t),t), \quad \text{for } 0 \leq t \leq T.$$

*Then*

$$u(x,t) \leq \quad \text{(or } \geq \text{)}\ g \quad \text{in } E$$

# Main Results




Cases:

I $b \neq 0$ ($b > 0,\ \beta > 0$ or $b < 0,\ \beta \geq 1$) and $m \geq 1$ (Figure above)

II $b = 0$

Case I.1) Suppose $\alpha < 2/\big(m - \min\{1,\beta\}\big)$. Lemma 2 $\Rightarrow$

$$u(x,t) \sim f(\rho) t^{\alpha/\big(2-\alpha(m-1)\big)},\ t \downarrow 0; \quad x = \xi_\rho(t) = \rho t^{1/\big(2-\alpha(m-1)\big)}$$

$$\liminf_{t\to 0^+} \eta(t) t^{1/\big(\alpha(m-1)-2\big)} \geq \xi_*$$

For $\epsilon > 0$, let $u_\epsilon$ solve

$$\begin{cases} u_t - (u^m)_{xx} = 0, \quad x \in \mathbb{R},\ 0 < t < T, \\ u_0(x) = (C+\epsilon)(-x)_+^\alpha \end{cases}$$

Let $b > 0$. $u_\epsilon$ is a supersolution of (1).

$$\eta(t) \leq (C+\epsilon)^{\frac{m-1}{2-\alpha(m-1)}} \xi'_* t^{1/\big(2-\alpha(m-1)\big)},\ 0 \leq t \leq \delta$$

$$\limsup_{t\to 0^+} \eta(t) t^{1/\big(\alpha(m-1)-2\big)} \leq \xi_*$$

Let $b < 0$ and $\beta \geq 1$.

$$\bar{u}_\epsilon(x,t) = \exp(-bt) u_\epsilon \left( x, \Big(b(1-m)\Big)^{-1} \Big[ \exp(b(1-m)t) - 1 \Big] \right)$$

Solves

$$\begin{cases} u_t - (u^m)_{xx} + bu = 0, & x \in \mathbb{R},\ 0 < t < T, \\ u_0(x) = (C+\epsilon)(-x)_+^\alpha \end{cases}$$

$$u_\epsilon < 1 \text{ in } B = \{(x,t) : x \geq x_\epsilon,\ 0 < t \leq \delta\}$$

$\bar{u}_\epsilon$ is a supersolution of (1) in $B$.

$$\eta(t) \leq (C+\epsilon)^{\frac{m-1}{2-\alpha(m-1)}} \xi_*' \Big\{ \Big(b(1-m)\Big)^{-1} \cdot$$

$$\cdot \Big[ \exp(b(1-m)t) - 1 \Big] \Big\}^{1/\big(2-\alpha(m-1)\big)},\ 0 \leq t \leq \delta$$

$$\limsup_{t \to 0^+} \eta(t) t^{1/\big(\alpha(m-1)-2\big)} \leq \xi_*$$

$$\limsup_{t \to 0^+} \eta(t) t^{1/\big(\alpha(m-1)-2\big)} \leq \xi_*$$

$$\liminf_{t \to 0^+} \eta(t) t^{1/\big(\alpha(m-1)-2\big)} \geq \xi_*$$

$$\eta(t) \sim \xi_* t^{1/\big(2-\alpha(m-1)\big)}, \; t \to 0^+$$

Case I.1 is proved

Case I.2) Consider $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha = 2/(m-\beta)$. Assume first $u_0 = C(-x)_+^\alpha$.

1. If $m + \beta = 2$,

$$u(x,t) = C(\zeta_* t - x)_+^{1/(1-\beta)}, \quad \zeta_* = b(1-\beta)C^{\beta-1}\big((C/C_*)^{m-\beta} - 1\big)$$

2. If $m + \beta \neq 2$, Lemma 3 $\Rightarrow$

$$u(x,t) = t^{1/(1-\beta)} f_1(\zeta), \quad \zeta = xt^{-\frac{m-\beta}{2(1-\beta)}}$$

Suppose $C>C_*$.

$$g(x,t)=t^{1/(1-\beta)}f_1(\zeta),\quad \zeta=xt^{-\frac{m-\beta}{2(1-\beta)}}$$

$$Lg=t^{\beta/(1-\beta)}L^0f_1$$

$$L^0f_1=\frac{1}{1-\beta}f_1-(f_1^m)''-\frac{m-\beta}{2(1-\beta)}\zeta f_1'+bf_1^{\beta}$$

$$f_1(\zeta)=C_0(\zeta_0-\zeta)_+^{\gamma_0}$$

$\gamma_0=2/(m-\beta)\Rightarrow$

$$L^0f_1=bC_0^{\beta}(\zeta_0-\zeta)^{\frac{2\beta}{m-\beta}}\left\{1-\left(\frac{C_0}{C_*}\right)^{m-\beta}+\frac{C_0^{1-\beta}}{b(1-\beta)}\zeta_0(\zeta_0-\zeta)^{\frac{2-m-\beta}{m-\beta}}\right\}$$

By choosing $C_0$, $\zeta_0$, when $m + \beta > 2$ or $1 \leq m < 2 - \beta$,

$$L^0 f_1 \geq 0, \ 0 \leq \zeta \leq \zeta_2$$

$$Lg \geq 0 \text{ for } 0 < x < \zeta_2 t^{\frac{m-\beta}{2(1-\beta)}}, \ t > 0$$

$$Lg = 0 \text{ for } x > \zeta_2 t^{\frac{m-\beta}{2(1-\beta)}}$$

$g$ is a supersolution for $x, t > 0$

$$g(x,0) = u(x,0) = 0, \ \text{for } 0 \leq x < +\infty$$

$$g(0,t) = u(0,t), \ \text{for } 0 \leq t < +\infty$$

From theorem 10,

$$u \leq C_2 t^{\frac{1}{1-\beta}} (\zeta_2 - \zeta)_+^{\frac{2}{m-\beta}}$$

Upper estimation is proved for $C > C_*$.

By choosing $C_0$, $\zeta_0$, when $m+\beta>2$ or $1\leq m<2-\beta$,

$$L^0 f_1 \leq 0,\ 0\leq \zeta \leq \zeta_1$$

$$Lg \leq 0 \text{ for } 0<x<\zeta_1 t^{\frac{m-\beta}{2(1-\beta)}},\ t>0$$
$$Lg = 0 \text{ for } x>\zeta_1 t^{\frac{m-\beta}{2(1-\beta)}}$$

$g$ is a subsolution for $x, t>0$.
As before, from theorem 10,

$$C_1 t^{\frac{1}{1-\beta}} (\zeta_1-\zeta)_+^{\mu} \leq u$$

$$\mu = \begin{cases} (m-1)^{-1}, & m+\beta>2 \\ 2(m-\beta)^{-1}, & 1\leq m<2-\beta \end{cases}$$

Lower estimation is proved when $C>C_*$

Suppose $m+\beta>2$ and $0<C<C_*$. Consider

$$g(x,t)=\Big[C^{1-\beta}(-x)_+^{\frac{2(1-\beta)}{m-\beta}}-b(1-\beta)(1-\gamma)t\Big]_+^{\frac{1}{1-\beta}},\quad x\in\mathbb{R},\ t>0$$

Estimate $Lg$ in

$$M=\{(x,t):\ -\infty<x<\mu_\gamma(t),\ t>0\},$$
$$\mu_\gamma(t)=-\left[b(1-\beta)(1-\gamma)C^{\beta-1}t\right]^{\frac{m-\beta}{2(1-\beta)}}$$

$Lg=bg^{\beta}S$ where

$$S=\gamma-2mb^{-1}(m-\beta)^{-2}(2-m-\beta)C^{m-\beta}\left[1-\frac{b(1-\beta)(1-\gamma)t}{C^{1-\beta}(-x)^{\frac{2(1-\beta)}{m-\beta}}}\right]^{\frac{m-1}{1-\beta}}-$$
$$-4mb^{-1}(m-\beta)^{-2}(m+\beta-1)C^{m-\beta}\left[1-\frac{b(1-\beta)(1-\gamma)t}{C^{1-\beta}(-x)^{\frac{2(1-\beta)}{m-\beta}}}\right]^{\frac{m+\beta-2}{1-\beta}}$$

# Proof of Main Results

Part I; 2) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha = 2/(m-\beta)$, $u_0 = C(-x)_+^\alpha$, $m + \beta > 2$, $0 < C < C_*$

$$S\Big|_{t=0} = \gamma - \left(\frac{C}{C_*}\right)^{m-\beta}, \quad S\Big|_{x=\mu_\gamma(t)} = \gamma$$

$$S_t \geq 0 \text{ in } M$$

$$\gamma - \left(\frac{C}{C_*}\right)^{m-\beta} \leq S \leq \gamma \text{ in } M$$

$$\gamma = (C/C_*)^{m-\beta} \ \big(\text{resp. } \gamma = 0\big) \ \Rightarrow \begin{array}{l} Lg \geq 0 \ (\text{resp. } Lg \leq 0), \ \text{in } M \\ Lg = 0 \text{ for } x > \mu_\gamma(t), \ t > 0 \end{array}$$

$$\left[C^{1-\beta}(-x)_+^{\frac{2(1-\beta)}{m-\beta}} - b(1-\beta)t\right]_+^{\frac{1}{1-\beta}} \leq u \leq$$

$$\left[C^{1-\beta}(-x)_+^{\frac{2(1-\beta)}{m-\beta}} - b(1-\beta)\left(1 - (C/C_*)^{m-\beta}\right)t\right]_+^{\frac{1}{1-\beta}}$$

# Proof of Main Results

Part I; 2) $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha = 2/(m-\beta)$, $u_0 = C(-x)_+^\alpha$, $1 \leq m < 2-\beta$, $0 < C < C_*$

Suppose $1 \leq m < 2-\beta$ and $0 < C < C_*$ Establish the rough estimate

$$\left[C^{1-\beta}(-x)_+^{\frac{2(1-\beta)}{m-\beta}} - b(1-\beta)\big(1-(C/C_*)^{m-\beta}\big)t\right]_+^{\frac{1}{1-\beta}} \leq u(x,t) \leq$$

$$\leq C(-x)_+^{\frac{2}{m-\beta}} \text{ for } x \in \mathbb{R},\ 0 \leq t < +\infty$$

To prove the left estimation, take $g$ with $\gamma = (C/C_*)^{m-\beta}$

$$g(x,t) = \left[C^{1-\beta}(-x)_+^{\frac{2(1-\beta)}{m-\beta}} - b(1-\beta)(1-\gamma)t\right]_+^{\frac{1}{1-\beta}}, \quad x \in \mathbb{R},\ t > 0$$

and derive similarly

$$Lg \leq 0, \text{ in } M; \quad Lg = 0 \text{ for } x > \mu_\gamma(t),\ t > 0$$

To prove the right-hand side it is enough to observe that

$$Lu_0 = bu_0^\beta\big(1-(C/C_*)^{m-\beta}\big) \geq 0 \text{ for } x \in \mathbb{R},\ t \geq 0$$

To prove an accurate estimation consider

$$g(x,t) = C_0 \left(-\zeta_0 t^{\frac{m-\beta}{2(1-\beta)}} - x\right)_+^{\frac{2}{m-\beta}} \text{ in } G_l$$

$$G_l = \left\{(x,t) : \ \zeta(t) = -l t^{\frac{m-\beta}{2(1-\beta)}} < x < +\infty, \ 0 < t < +\infty\right\}$$

$$Lg = bg^{\beta} S, \quad S = 1 - (C_0/C_*)^{m-\beta} - \big(b(1-\beta)\big)^{-1} C_0^{1-\beta} \zeta_0 t^{\frac{m+\beta-2}{2(1-\beta)}} \times$$
$$\times \left(-\zeta_0 t^{\frac{m-\beta}{2(1-\beta)}} - x\right)^{\frac{2-m-\beta}{m-\beta}}$$

$$\text{in } G_l^+ = \left\{(x,t) : \ \zeta(t) < x < -\zeta_0 t^{\frac{m-\beta}{2(1-\beta)}}, \ 0 < t < +\infty\right\}$$

If $C_0 = C_*$ then $Lg \leq 0$ in $G_l^+$; $Lg = 0$ in $G_l \setminus \bar{G}_l^+$

Choose $\zeta_0$, $l$, and use the rough estimation to show

$$g(\zeta(t),t) = C_*(l_0 - \zeta_3)^{\frac{2}{m-\beta}} t^{\frac{1}{1-\beta}} \leq u(\zeta(t),t),\ t \geq 0$$
$$g(x,0) = u(x,0) = 0,\ 0 \leq x \leq x_0$$
$$g(x_0,t) = u(x_0,0) = 0,\ t \geq 0$$

Apply comparison thm 10 in $G'_{l_0} = G_{l_0} \cap \{x < x_0\}$ to derive

$$C_* \left( -\zeta_3 t^{\frac{m-\beta}{2(1-\beta)}} - x \right)_+^{\frac{2}{m-\beta}} \leq u$$

Derive the upper estimation: $S_x \geq 0$ for $\zeta(t) < x < -\zeta_0 t^{\frac{m-\beta}{2(1-\beta)}}$

$$S \geq S\Big|_{x=\zeta(t)} = 1 - (C_0/C_*)^{m-\beta} - \big(b(1-\beta)\big)^{-1} C_0^{1-\beta} \zeta_0 (l-\zeta_0)^{\frac{2-m-\beta}{m-\beta}}$$

Choose $C_0,\zeta_0,l$ to find $S\Big|_{x=\zeta(t)}=0$

$$Lg\geq 0 \text{ in } G_{l_1}^+,\ Lg=0 \text{ in } G_{l_1}\setminus \bar{G}_{l_1}^+$$
$$u(\zeta(t),t)\leq C l_1^{\frac{2}{m-\beta}}=C_3(l_1-\zeta_4)^{\frac{2}{m-\beta}}t^{\frac{1}{1-\beta}}=g(\zeta(t),t),\ t\geq 0$$

Apply comparison thm 10 in $G'_{l_1}$ to derive

$$u\leq C_3\left(-\zeta_4 t^{\frac{m-\beta}{2(1-\beta)}}-x\right)_+^{\frac{2}{m-\beta}},\ 0\leq t<+\infty$$

$$\zeta_1 t^{\frac{m-\beta}{2(1-\beta)}}\leq \eta(t)\leq \zeta_2 t^{\frac{m-\beta}{2(1-\beta)}},\quad 0\leq t<+\infty$$

$$\eta(t)=\zeta_* t^{\frac{m-\beta}{2(1-\beta)}},\quad 0\leq t<+\infty$$

Suppose $u_0 \sim C(-x)_+^\alpha$, $C \neq C_*$ Similarly to the proof of lemma 1,

$$\eta(t) \sim \zeta_* t^{\frac{m-\beta}{2(1-\beta)}} \text{ as } t \to 0^+$$

$$\rho < \zeta_* \Rightarrow u(x,t) \sim f_1(\rho) t^{1/(1-\beta)} \text{ for } x = \rho t^{\frac{m-\beta}{2(1-\beta)}},\ t \to 0^+$$

Case I.2 is proved.

Case I.3) Suppose $b > 0$, $0 < \beta < 1$, $m \geq 1$, $\alpha > 2/(m-\beta)$.
Let $\epsilon > 0$; $u_0 \sim C(-x)_+^\alpha \Rightarrow$

$$(C-\epsilon/2)\,(-x)_+^\alpha \leq u_0(x) \leq (C+\epsilon/2)(-x)_+^\alpha,\ x \geq x_\epsilon$$

$$g_\epsilon(x,t) = \Big[(C+\epsilon)^{1-\beta}(-x)_+^{\alpha(1-\beta)} - (1-\beta)(1-\epsilon)t\Big]_+^{1/(1-\beta)}$$

Estimate $Lg$ in

$$M_1 = \Big\{(x,t) : x_\epsilon < x < \eta_{l(\epsilon)}(t),\ 0 < t \leq \delta_1\Big\}$$
$$\eta_l(t) = -lt^{1/\alpha(1-\beta)},\ l(\epsilon) = (C+\epsilon)^{-1/\alpha}\left[b(1-\beta)(1-\epsilon)\right]^{1/\alpha(1-\beta)}$$
$$Lg_\epsilon = bg_\epsilon^\beta\{\epsilon + S\}$$
$$S = -b^{-1}m\alpha(C+\epsilon)^{m-\beta}(-x)^{\alpha(m-\beta)-2}\{g|x|^{-\alpha}/(C+\epsilon)\}^{m+\beta-2}S_1$$
$$S_1 = \Big\{\big(\alpha(1-\beta)-1\big)\left[g|x|^{-\alpha}/(C+\epsilon)\right]^{1-\beta} + \alpha(m+\beta-1)\Big\}$$

Suppose $m+\beta \geq 2$. Then $|S| < \epsilon/2$ in $M_1$ so

$$Lg_\epsilon > b(\epsilon/2)g_\epsilon^\beta \text{ (resp. } Lg_{-\epsilon} < -b(\epsilon/2)g_{-\epsilon}^\beta) \text{ in } M_1$$
$$Lg_{\pm\epsilon} = 0 \text{ for } x > \eta_{l(\pm\epsilon)}(t),\ 0 < t \leq \delta_1$$
$$g_\epsilon(x,0) \geq u_0(x) \text{ (resp. } g_{-\epsilon}(x,0) \leq u_0(x)),\ x \geq x_\epsilon$$

Choose $\delta \in (0, \delta_1]$ such that

$$g_\epsilon(x_\epsilon, t) \geq u(x_\epsilon, t) \text{ (resp. } g_{-\epsilon}(x_\epsilon, t) \leq u(x_\epsilon, t)),\ 0 \leq t \leq \delta$$

$$g_{-\epsilon} \leq u \leq g_\epsilon, \quad x \geq x_\epsilon,\ 0 \leq t \leq \delta$$
$$\eta_{l(-\epsilon)}(t) \leq \eta(t) \leq \eta_{l(\epsilon)}(t), \quad 0 \leq t \leq \delta$$

$$\eta(t) \sim -l_* t^{1/\alpha(1-\beta)},\ t \to 0^+, \quad l_* = C^{-1/\alpha}\left(b(1-\beta)\right)^{1/(\alpha(1-\beta)}$$

$$u(x,t) \sim \left[C^{1-\beta}(-x)_+^{\alpha(1-\beta)} - b(1-\beta)t\right]^{1/(1-\beta)} \text{ as } t \to 0^+$$

Suppose $1 \leq m < 2-\beta$. Lower est. is similar. Prove an upper estimation:

$$g(x,t) = C_6 \left(-\zeta_5 t^{\frac{1}{\alpha(1-\beta)}} - x\right)_+^\alpha \text{ in } G_{l,\delta},$$
$$G_{l,\delta} = \{(x,t) : \eta_l(t) < x < +\infty,\ 0 < t < \delta\}$$

For $l > l_*$, $\epsilon > 0$, $\exists \delta = \delta(l,\epsilon)$ s.t.

$$u(\eta_l(t),t) \leq \left[C^{1-\beta} l^{\alpha(1-\beta)} - b(1-\beta)(1-\epsilon)\right]^{\frac{1}{1-\beta}} t^{\frac{1}{1-\beta}}, \quad 0 \leq t \leq \delta.$$

$$Lg \ \text{ in } G^+_{l,\delta} = \left\{(x,t) : \eta_l(t) < x < -\zeta_5 t^{\frac{1}{\alpha(1-\beta)}},\ 0 < t < \delta\right\}:$$

$$Lg = bg^\beta S, \quad S = 1 - \big(b(1-\beta)\big)^{-1} C_6^{1/\alpha} \zeta_5 \{g t^{1/(\beta-1)}\}^{1-\beta-1/\alpha} -$$
$$-b^{-1}\alpha m(\alpha m - 1) C_6^{2/\alpha} g^{m-\beta-2/\alpha}$$

$$S_x \geq 0 \text{ in } G^+_{l,\delta}$$

$$S \geq S\Big|_{x=\eta_l(t)} \geq \epsilon - b^{-1} C_6^{m-\beta} \alpha m(\alpha m - 1) \left\{(l-\zeta_5) t^{1/\alpha(1-\beta)}\right\}^{\alpha(m-\beta)-2} \text{ in } G^+_{l,\delta}$$

Choose $\delta$ s.t. $Lg \geq b(\epsilon/2) g^\beta$ in $G^+_{l,\delta}$

$$Lg = 0 \text{ in } G'_{l,\delta} \setminus \bar{G}^+_{l,\delta}$$

$$u(\eta_l(t), t) \leq \left[C^{1-\beta} l^{\alpha(1-\beta)} - b(1-\beta)(1-\epsilon)\right]^{\frac{1}{1-\beta}} t^{\frac{1}{1-\beta}}$$
$$= C_6 (l-\zeta_5)^\alpha t^{\frac{1}{\alpha(1-\beta)}} = g(\eta_l(t), t), \quad 0 \leq t \leq \delta$$

$$u(x_0, t) = g(x_0, t) = 0, \quad 0 \leq t \leq \delta$$
$$u(x, 0) = g(x, 0) = 0, \quad 0 \leq x \leq x_0$$

Comparison theorem 10 $\Rightarrow$

$$u(x,t) \leq C_6 \left(-\zeta_5 t^{\frac{1}{\alpha(1-\beta)}} - x\right)_+^{\alpha} \text{ in } \bar{G}_{l,\delta}$$

$$-lt^{1/\alpha(1-\beta)} \leq \eta(t) \leq -\zeta_5 t^{1/\alpha(1-\beta)}, \quad 0 \leq t \leq \delta$$

where $\zeta_5 = \left(\frac{l_*}{l}\right)^{\alpha(1-\beta)}(1-\epsilon)l$.

$$\eta(t) \sim -l_* t^{1/\alpha(1-\beta)},\ t \to 0^+, \quad l_* = C^{-1/\alpha}\left(b(1-\beta)\right)^{1/(\alpha(1-\beta)}$$

$$u(x,t) \sim \left[C^{1-\beta}(-x)_+^{\alpha(1-\beta)} - b(1-\beta)t\right]^{1/(1-\beta)} \text{ as } t \to 0^+$$

along $x = \eta_l(t) = -lt^{1/\alpha(1-\beta)}, l > l_*$. Case I.3 is proved.

Case 4a) is immediate.
Case 4b) Suppose $\beta = 1,\ \alpha > 2/(m-1)$.

$$\forall 0 < \epsilon \ll 1 \Rightarrow (C - \epsilon/2)\,(-x)_+^{\alpha} \le u_0(x) \le (C + \epsilon/2)(-x)_+^{\alpha},\ x \ge x_\epsilon$$

$$g(x,t) = (C - \epsilon)(-x)_+^{\alpha} \exp(-bt)$$

$$Lg \le 0 \text{ for } x_\epsilon < x < 0,\ t > 0; \quad Lg = 0 \text{ for } x, t > 0$$

$$\text{Choose } \delta \text{ s.t. } g(x_\epsilon, t) \le u(x_\epsilon, t) \text{ for } 0 \le t \le \delta_\epsilon$$

$$(C - \epsilon)(-x)_+^{\alpha} \exp(-bt) \le u(x,t) \text{ for } x \ge x_\epsilon,\ 0 \le t \le \delta_\epsilon$$

Lower Estimate is Proved

# Proof of Main Results

Part I; 4b) $\beta = 1$, $\alpha > 2/(m-1)$

$$g(x,t) = (C+\epsilon)(-x)_+^{\alpha} \exp(-bt) \cdot$$
$$\cdot \Big[1 - \epsilon\big(b(m-1)\big)^{-1}\big(1 - \exp(-b(m-1)t)\big)\Big]^{1/(1-m)}$$

$$|x_\epsilon| \ll 1 \Rightarrow Lg \geq 0 \text{ for } x_\epsilon < x < 0,\ t > 0; \quad Lg = 0 \text{ for } x, t > 0$$

$$u(x,t) \leq (C+\epsilon)(-x)_+^{\alpha} \exp(-bt) \cdot$$
$$\cdot \Big[1 - \epsilon\big(b(m-1)\big)^{-1}\big(1 - \exp(-b(m-1)t)\big)\Big]^{1/(1-m)}$$

Case I.4b is proved. (4c and 4d are similar.)

Part II ($b=0$), Case 1). Suppose $m>1$, $0<\alpha<2/(m-1)$.

$$u_0(x)=C(-x)_+^{\alpha} \Rightarrow u(x,t)=t^{\frac{\alpha}{2+\alpha(1-m)}}f(\xi), \quad \xi=xt^{-\frac{1}{2+\alpha(1-m)}}$$

$$\eta(t)=\xi_* t^{\frac{1}{2+\alpha(1-m)}} \text{ for } 0\le t<+\infty$$

Consider

$$g(x,t)=t^{\alpha/\left(2-\alpha(m-1)\right)}f(\xi)$$

$$Lg=t^{(m\alpha-2)/(2-\alpha(m-1)}L_t f$$

$$L_t f=\frac{\alpha}{2-\alpha(m-1)}f-\frac{1}{2-\alpha(m-1)}\xi f'-(f^m)''$$

$$f(\xi)=C_0(\xi_0-\xi)_+^{1/(m-1)}$$

$$L_t f = \left(2-\alpha(m-1)\right)^{-1}(m-1)^{-1}C_0(\xi_0-\xi)^{\frac{2-m}{m-1}}R(\xi)$$

for $0\leq\xi<\xi_0,\ t>0$

$$R(\xi)=\alpha(m-1)\xi_0+\big(1-\alpha(m-1)\big)\xi-(m-1)^{-1}m\big(2-\alpha(m-1)\big)C_0^{m-1}$$

Choosing $C_0,\xi_0$,

$$R(\xi)\geq\nu_\alpha\xi_4-(m-1)m\big(2-\alpha(m-1)\big)C_5^{m-1}=0$$

$$Lg\geq 0 \text{ for } 0<x<\xi_4 t^{1/\left(2-\alpha(m-1)\right)},\ t>0$$

$$Lg=0 \text{ for } x>\xi_4 t^{1/\left(2-\alpha(m-1)\right)},\ t>0$$

$$u(0,t)=g(0,t),\ t>0; u(x,0)=g(x,0),\ x\geq 0$$

$$u(x,t)\leq C_5 t^{\frac{\alpha}{2+\alpha(1-m)}}\left(\xi_4-\xi\right)_+^{\frac{1}{m-1}}$$

$$C_4 t^{\frac{\alpha}{2+\alpha(1-m)}}\left(\xi_3-\xi\right)_+^{\frac{1}{m-1}}\leq u(x,t)$$

Remaining results, and case $u_0 \sim C(-x)_+^\alpha$ are established using lemma 1.
Part II ($b = 0$), Case 2) Suppose $m > 1$, $\alpha = 2/(m-1)$
$u_0(x) = C(-x)_+^\alpha \Rightarrow$

$$u_C(x,t) = C(-x)_+^\alpha \left[1 - (C/\bar{C})^{m-1}(m-1)t\right]^{1/(1-m)}, \ x \in \mathbb{R}, \ 0 \le t < T$$

$u_0(x) \sim C(-x)_+^\alpha \Rightarrow$

$$u_{C-\epsilon} \le u \le u_{C+\epsilon}$$

Part II ($b = 0$), Case 3) Suppose $m > 1$, $\alpha > 2/(m-1)$
For all $\epsilon > 0$ $\exists x_\epsilon < 0, \ \delta_\epsilon > 0$ s.t.

$$(C-\epsilon)(-x)_+^\alpha \le u \le (C+\epsilon)(-x)_+^\alpha (1-\epsilon t)^{1/(1-m)}, \quad x_\epsilon \le x, \ 0 \le t \le \delta_\epsilon$$

□

Cauchy problem for the Reaction-Diffusion-Convection equation:

$$u_t - (u^m)_{xx} + bu^\beta + c(u^\gamma)_x = 0, \quad x \in \mathbb{R},\ 0 < t < T, \tag{13}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R} \tag{14}$$

where $m > 0$, $b, c \in \mathbb{R}$, $\beta, \gamma > 0$, $0 < T \leq +\infty$, $u_0 \geq 0$, $u_0 \in C(\mathbb{R})$.

$$\eta(t) = \sup\{x : u(x,t) > 0\}, \quad \eta(0) = 0$$

$$u_0(x) \sim C(-x)_+^\alpha, \quad \text{as } x \to 0-, \quad \text{for some } C > 0, \alpha > 0. \tag{15}$$

$$u_0(x) = C(-x)_+^\alpha, \quad x \in \mathbb{R} \tag{16}$$

*Does interface expand, shrink or remain stationary? Find the short-time behavior of the interface function $\eta(t)$ and $u(x,t)$ near $x = \eta(t)$.*
The **Answer** depends on $m, \beta, \gamma, b, c, C, \alpha$. Is it possible to give a full classification?

Nonhomogeneous Reaction-Diffusion-Convection equation:

$$u_t - (a(x,t)u^m)_{xx} + b(x,t)u^\beta + c(x,t)(u^\gamma)_x = 0, \quad x \in \mathbb{R},\ 0 < t < T, \tag{17}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R} \tag{18}$$

where $m, \beta, \gamma > 0$, $a, b, c \in C(\mathbb{R})$.
Reaction-Diffusion-Convection equations with non-standard growth condition:

$$u_t - (u^{m(x)})_{xx} + bu^{\beta(x)} + c(u^{\gamma(x)})_x = 0, \quad x \in \mathbb{R},\ 0 < t < T, \tag{19}$$

where $m, \beta, \gamma \in C(\mathbb{R})$.

$$\eta(t) = \sup\{x : u(x,t) > 0\}, \quad \eta(0) = 0$$

$$u_0(x) \sim C(-x)_+^\alpha, \quad \text{as } x \to 0-, \quad \text{for some } C > 0, \alpha > 0. \tag{20}$$

*Find the short-time behavior of the interface function $\eta(t)$ and $u(x,t)$ near $x = \eta(t)$.*

Cauchy problem for the Reaction-Diffusion equation:

$$u_t - \Delta(u^m) + bu^\beta = 0, \quad x \in \mathbb{R}^n,\ 0 < t < T, \tag{21}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n \tag{22}$$

where $m > 0$, $b \in \mathbb{R}$, $\beta > 0$, $0 < T \leq +\infty$, $u_0 \geq 0$, $u_0 \in C(\mathbb{R})$.
Let $\operatorname{supp}(u_0)$ is compact, $\partial \operatorname{supp}(u_0)$ possibly has a corner or cusp near $0$.
*Given asymptotic behaviour of $u_0$ near $\partial \operatorname{supp}(u_0)$ find the evolution of the interfaces with possible singularities and $u(x,t)$ near the interface.*

# Open Problems



Cauchy problem for the Reaction-Diffusion system:

$$u_t = \Delta(u^{m_1}) + bu^{\beta_1}v^{\gamma_1}, \quad x \in \mathbb{R}^n,\ 0 < t < T, \tag{23}$$

$$v_t = \Delta(v^{m_2}) - cu^{\beta_2}v^{\gamma_2}, \quad x \in \mathbb{R}^n,\ 0 < t < T, \tag{24}$$

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x) \quad x \in \mathbb{R}^n \tag{25}$$

$\eta(t) = \sup\{x : u(x,t) > 0\},\ \zeta(t) = \sup\{x : v(x,t) > 0\} \quad \eta(0) = \zeta(0) = 0,$

*Find the short-time behavior of the interface functions* $\eta(t), \zeta(t)$ *and* $u(x,t), v(x,t)$ *near* $x = \eta(t), x = \zeta(t)$*.*