Evolution of Interfaces for the Reaction-Diffusion Equation

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FIT Colloquium

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Introduction

Cauchy problem for the Reaction-Diffusion equation:

$$\mathcal{L}u \equiv u_t - (u^m)_{xx} + bu^\beta = 0, x \in \mathbb{R}, 0 < t < T, \quad (1)$$

$$u(x,0) = u_0(x), x \in \mathbb{R} \quad (2)$$

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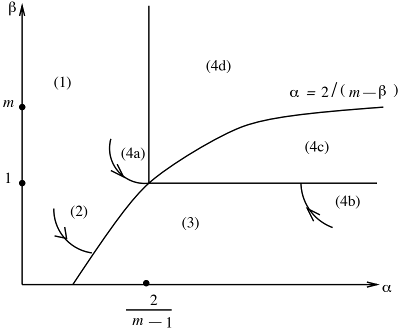
Barenblatt's problem: *Does interface expand, shrink or remain stationary? Find the short-time behavior of the interface function $\eta(t)$ and $u(x,t)$ near $x = \eta(t)$.*
The **Answer** depends on m, β, b, C, α. Is it possible to give a full classification?

Full Classification

U. G. Abdulla, Reaction-diffusion in irregular domains, J. Differential Equations, 164, 2000, 321-354.
U. G. Abdulla and J. R. King, Interface development and local solutions to reaction-diffusion equations, SIAM J. Math. Anal., 32, 2, 2000, 235-260.
U. G. Abdulla, Evolution of interfaces and explicit asymptotics at infinity for the fast diffusion equation with absorption, Nonlinear Analysis, 50, 4, 2002, 541-560.



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If u_0 satisfies (3) then $\eta(t) \sim \xi_* t^{\frac{1}{2+\alpha(1-m)}}$ *as* $t \downarrow 0$ *and* $\forall \rho < \xi_*$

$$u(x,t) \sim f(\rho) t^{\frac{\alpha}{2+\alpha(1-m)}} \quad \text{as } t \downarrow 0 \quad \text{along } x = \xi_\rho(t) = \rho t^{\frac{1}{2+\alpha(1-m)}}$$

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Choose $k = t^{\frac{\alpha}{2-\alpha(m-1)}} \Rightarrow u(x,t) = t^{\frac{\alpha}{2+\alpha(1-m)}}u(xt^{-\frac{1}{2+\alpha(1-m)}}, 1)$
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Choose $\tau = C^{m-1}t, w(x,\tau) = v(x, C^{1-m}\tau) \Rightarrow w$ solves (1), (2), (4) with $C = 1$. Hence $u(x,t) = Cw(x, C^{m-1}t)$. From (5) $\Rightarrow$

$$u(x,t) = kw(C^{\frac{1}{\alpha}}k^{-\frac{1}{\alpha}}x, C^{\frac{2}{\alpha}}k^{\frac{\alpha(m-1)-2}{\alpha}}t) \tag{6}$$

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$$\epsilon \downarrow 0\ \Rightarrow u(\xi_\rho(t),t) \sim f(\rho;C)t^{\frac{\alpha}{2+\alpha(1-m)}} \quad \text{as } t\downarrow 0;\ \forall \rho < \xi_*$$

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Take in $u_{-\epsilon} \le u \le u_\epsilon \ x = \xi_\rho(t)$, multiply by $t^{-\frac{\alpha}{2+\alpha(1-m)}} \Rightarrow$

$$u_{-\epsilon}(\xi_\rho(t), t) t^{-\frac{\alpha}{2+\alpha(1-m}} \le u(\xi_\rho(t), t) t^{-\frac{\alpha}{2+\alpha(1-m)}} \le u_\epsilon(\xi_\rho(t), t) t^{-\frac{\alpha}{2+\alpha(1-m)}}$$

$$f(\rho; C-\epsilon) \le u(\xi_\rho(t), t) t^{-\frac{\alpha}{2+\alpha(1-m)}} \le f(\rho; C+\epsilon), \ 0 \le t \le \delta$$

$$f(\rho; C-\epsilon) \le \liminf_{t \downarrow 0} u(\xi_\rho(t), t) t^{\frac{\alpha}{\alpha(m-1)-2}} \le \limsup_{t \downarrow 0} \cdots \le f(\rho; C+\epsilon)$$

$$\epsilon \downarrow 0 \ \Rightarrow u(\xi_\rho(t), t) \sim f(\rho; C) t^{\frac{\alpha}{2+\alpha(1-m)}} \quad \text{as } t \downarrow 0; \ \forall \rho < \xi_*$$

$$\xi_*(C-\epsilon) t^{\frac{1}{2+\alpha(1-m)}} \le \eta(t) \le \xi_*(C+\epsilon) t^{\frac{1}{2+\alpha(1-m)}}, \ 0 \le t \le \delta$$

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$$\eta\sim\xi_*t^{\frac{1}{2+\alpha(1-m)}},\quad \text{as } t\downarrow 0;\ \xi_*=C^{\frac{m-1}{2+\alpha(1-m)}}\xi'_*$$

Lemma 1 is proved.

Summary of Lemma 1

Consider

$$\begin{cases} u_t = (u^m)_{xx}, \ x \in \mathbb{R}, \ t > 0 \\ u(x,0) = C(-x)_+^{\alpha}, \ x \in \mathbb{R} \end{cases} \tag{7}$$

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In fact, $f(\xi) = u(\xi, 1)$.

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$$f(\rho) = C^{\frac{2}{2-\alpha(m-1)}} f_0(C^{\frac{m-1}{2-\alpha(m-1)}} \rho)$$

f_0 solves (8) with $C = 1$; $f_0(\xi) = w(\xi, 1)$, $w(\xi, 1)$ solves (7) with $C = 1$.

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If $u_0(x) \sim C(-x)_+^\alpha \Rightarrow \eta(t) \sim \xi_* t^{\frac{1}{2+\alpha(1-m)}}$ as $t \downarrow 0$ and $\forall \rho < \xi_*$

$$u(x,t) \sim f(\rho) t^{\frac{\alpha}{2+\alpha(1-m)}} \quad \text{as } t \downarrow 0 \quad \text{along } x = \xi_\rho(t) = \rho t^{\frac{1}{2+\alpha(1-m)}}$$

Lemma 2

Let u solve

$$u_t = (u^m)_{xx} - bu^\beta,\ x \in \mathbb{R},\ t > 0 \tag{9}$$

$$u(x,0) = u_0(x),\ x \in \mathbb{R} \tag{10}$$

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a) $b > 0$, $0 < \beta < 1 < m$, $0 < \alpha < 2/(m - \beta)$

b) $b \neq 0$, $\beta \geq 1$, $m > 1$, $0 < \alpha < 2/(m - 1)$

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Then

$$u(x,t) \sim f(\rho) t^{\frac{\alpha}{2-\alpha(m-1)}}, \quad x = \xi_\rho(t) = \rho t^{\frac{1}{2-\alpha(m-1)}}$$

Proof of Lemma 2

$u_0 \sim C(-x)_+^\alpha \Rightarrow \forall \epsilon > 0,\ \exists x_\epsilon < 0$

$$(C - \epsilon/2)(-x)_+^\alpha \le u_0(x) \le (C + \epsilon/2)(-x)_+^\alpha,\ x \ge x_\epsilon$$

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Scale u as in the diffusion case

$$u_k^{\pm\epsilon}(x,t) = k u_{\pm\epsilon}(k^{-\frac{1}{\alpha}} x, k^{\frac{\alpha(m-1)-2}{\alpha}} t),\ k \ge 0$$

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$$\alpha(m-\beta)-2<0 \ \Rightarrow \ \lim_{k\to\infty} u_k^{\pm\epsilon}(x,t)=v_{\pm\epsilon}(x,t),\ x\in\mathbb{R},\ t\geq 0 \quad (12)$$

$$v_{\pm\epsilon} \text{ solves} \begin{cases} v_t=(v^m)_{xx} \\ v\big|_{t=0}=(C\pm\epsilon)(-x)_+^{\alpha},\ T=+\infty \end{cases}$$

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By the previous lemma 1

$$v_{\pm\epsilon}(\xi_\rho(t),t) = f(\rho, C\pm\epsilon)t^{\frac{\alpha}{2-\alpha(m-1)}}$$

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$$\lim_{k\to\infty} ku_{\pm\epsilon}(k^{-\frac{1}{\alpha}}\xi_\rho(t),k^{\frac{\alpha(m-1)-2}{\alpha}}t)=f(\rho;C\pm\epsilon)t^{\frac{\alpha}{2-\alpha(m-1)}},\; t\geq 0 \quad (13)$$

Let $k^{\frac{\alpha(m-1)-2}{\alpha}}t=\tau$ so $k^{-\frac{1}{\alpha}}\xi_\rho(t)=\xi_\rho(\tau)$ so from (13),

$$\lim_{\substack{k\to+\infty \\ (\tau\to 0)}} \frac{ku_{\pm\epsilon}(k^{-\frac{1}{\alpha}}\xi_\rho(t),k^{\frac{\alpha(m-1)-2}{\alpha}}t)}{t^{\frac{\alpha}{2-\alpha(m-1)}}}=f(\rho;C\pm\epsilon)$$

Proof of Lemma 2

$$\lim_{\tau \to 0} \frac{u_{\pm\epsilon}(\xi_\rho(\tau), \tau)}{\tau^{\frac{\alpha}{2-\alpha(m-1)}}} = f(\rho; C \pm \epsilon)$$

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So

$$u_{\pm\epsilon}(\xi_\rho(\tau),\tau)\sim f(\rho;C\pm\epsilon)\tau^{\frac{\alpha}{2-\alpha(m-1)}},\ \tau\downarrow 0$$

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$$u_{\pm\epsilon}(\xi_\rho(\tau), \tau) \sim f(\rho; C \pm \epsilon)\tau^{\frac{\alpha}{2-\alpha(m-1)}}, \ \tau \downarrow 0$$

By comparison, from $u_{-\epsilon} \leq u \leq u_\epsilon$ it follows that

$$u(x,t) \sim f(\rho; C)t^{\frac{\alpha}{2-\alpha(m-1)}}, \ t \downarrow 0 \tag{14}$$

Lemma 2 is proved except in case b) with $b < 0$.

Proof of Lemma 2

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Lemma 2 is proved except in case b) with $b<0$. Let $b<0$, $\beta\geq 1$, $m>1$, $0<\alpha<2/(m-1)$ and suppose $u_{\pm\epsilon}$ solves

$$\begin{cases} u_t-(u^m)_{xx}+bu^\beta=0,\ |x|<|x_\epsilon|,\ 0<t\leq\delta \\ u(x,0)=(C\pm\epsilon)(-x)_+^\alpha \\ u(x_\epsilon,t)=(C\pm\epsilon)(-x_\epsilon)^\alpha,\ u(-x_\epsilon,t)=0,\ 0\leq t\leq\delta \end{cases} \tag{15}$$

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$[UA, JDE] \Rightarrow \exists\delta>0$ for which there is a local soln, and a comparison theorem implies

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Convergence of the $u_k^{\pm\epsilon}$: consider

$$g(x,t) = (C+1)(1+x^2)^{\alpha/2}(1-\nu t)^{\frac{1}{1-m}}, \ x \in \mathbb{R}, \ 0 \le t \le t_0 = \frac{\nu^{-1}}{2}$$

Proof of Lemma 2

Calculate

$$L_k g \equiv g_t - (g^m)_{xx} + bk^{\frac{\alpha(m-\beta)-2}{\alpha}} g^\beta = (C+1)(m-1)^{-1}(1+x^2)^{\frac{\alpha}{2}}(1-\nu t)^{\frac{m}{1-m}} S$$

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$$S = \nu - h(x) + b(m-1)(C+1)^{\beta-1} k^{\frac{\alpha(m-\beta)-2}{\alpha}} (1+x^2)^{\frac{\alpha(\beta-1)}{2}} (1-\nu t)^{\frac{\beta-m}{1-m}}$$

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$$\nu = h_* + 1, \; h_* = h(\alpha, m) = \max_{\mathbb{R}} h(x),$$

$$h(x) = (m-1)(C+1)^{m-1} \alpha m (1+x^2)^{\frac{\alpha(m-1)}{2}-2} \big(1 + (\alpha m - 1)x^2\big)$$

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Here $\frac{\alpha(m-1)-2}{2} < 0$ and $S \geq 1 + R$ where

$$R = O\Big(k^{\frac{\alpha(m-\beta)-2}{\alpha}} (k^{\frac{2}{\alpha}})^{\frac{\alpha(\beta-1)}{2}}\Big) = O\Big(k^{m-1-\frac{2}{\alpha}}\Big)$$

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$$L_k g \equiv g_t - (g^m)_{xx} + bk^{\frac{\alpha(m-\beta)-2}{\alpha}} g^\beta = (C+1)(m-1)^{-1}(1+x^2)^{\frac{\alpha}{2}}(1-\nu t)^{\frac{m}{1-m}} S$$

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So when $k \to \infty$ $S > 0 \Rightarrow L_k g \geq 0$ on $D^k_{o\epsilon} = D^k_\epsilon \cap \{0 < t \leq t_0\}$

Proof of Lemma 2

Calculate

$$L_k g \equiv g_t - (g^m)_{xx} + bk^{\frac{\alpha(m-\beta)-2}{\alpha}} g^\beta = (C+1)(m-1)^{-1}(1+x^2)^{\frac{\alpha}{2}}(1-\nu t)^{\frac{m}{1-m}} S$$

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Here $\frac{\alpha(m-1)-2}{2} < 0$ and $S \geq 1 + R$ where

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So when $k \to \infty$ $S > 0 \Rightarrow L_k g \geq 0$ on $D^k_{o\epsilon} = D^k_\epsilon \cap \{0 < t \leq t_0\}$

$$g(x,0) \geq u_k^{\pm\epsilon}(x,0)$$

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By comparison theorem,

$$0 \leq u_k^{\pm\epsilon}(x,t) \leq g(x,t) \text{ in } \overline{D_{o\epsilon}^k}$$

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For any compact $G \subset P = \{x \in \mathbb{R}, 0 < t \leq t_0\}$ $u_k^{\pm\epsilon}$ is uniformly bounded and ($[UA, JDE]$) uniformly Hölder continuous $\Rightarrow \exists v_{\pm\epsilon}$ s.t.

$$\lim_{k' \to +\infty} u_{k'}^{\pm\epsilon} = v_{\pm\epsilon}(x,t),\ (x,t) \in P$$

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$$u_t - (u^m)_{xx} + bk^{\frac{\alpha(m-\beta)-2}{\alpha}} u^\beta = 0$$

the exponent $\alpha(m-\beta) - 2 < 0$, it follows that $v_{\pm\epsilon}$ solves

$$u_t = (u^m)_{xx},\ x \in \mathbb{R},\ 0 < t \le t_0$$
$$u(x,0) = (C \pm \epsilon)(-x)_+^\alpha,\ x \in \mathbb{R}$$

The rest of the proof is similar to the previous case.

Proof of Lemma 2

By the previous lemma 1

$$v_{\pm\epsilon}(\xi_\rho(t), t) = f(\rho, C \pm \epsilon) t^{\frac{\alpha}{2-\alpha(m-1)}}$$

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Let $k^{\frac{\alpha(m-1)-2}{\alpha}}t = \tau$ so $k^{-\frac{1}{\alpha}}\xi_\rho(t) = \xi_\rho(\tau)$ so from (17),

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$$u_{\pm\epsilon}(\xi_\rho(\tau),\tau) \sim f(\rho; C\pm\epsilon)\tau^{\frac{\alpha}{2-\alpha(m-1)}},\ \tau\downarrow 0$$

By comparison, from $u_{-\epsilon}\leq u\leq u_\epsilon$ it follows that

$$u(x,t) \sim f(\rho; C)t^{\frac{\alpha}{2-\alpha(m-1)}},\ t\downarrow 0 \quad \text{along } x = \xi_\rho(t) = \rho t^{\frac{1}{2+\alpha(1-m)}}$$

Lemma is proved.

Lemma 3

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Let u solve

$$u_t = (u^m)_{xx} - bu^\beta,\ x \in \mathbb{R},\ t > 0 \tag{18}$$

$$u(x,0) = C(-x)_+^\alpha,\ x \in \mathbb{R},\ \alpha = 2/(m-\beta) \tag{19}$$

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Where $b > 0$, $0 < \beta < 1$, $m \geq 1$. Then

$$u(x,t) = t^{\frac{1}{1-\beta}} f_1(\zeta),\ \zeta = \frac{x}{t^{\frac{m-\beta}{2(1-\beta)}}}$$

Proof of Lemma 3

Choose $v(x,t) = ku(k^{\gamma_1}x, k^{\gamma_2}t)$ so

$$v_t - (v^m)_{xx} + bv^\beta = k^{1+\gamma_2}u_t - k^{m+2\gamma_1}(u^m)_{xx} + bk^\beta u^\beta$$

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Hence $f(\zeta) = u(\zeta, 1)$.

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When $x < 0$, $\lim_{t\to 0} u(x,t) = C(-x)_+^{\frac{2}{m-\beta}}$, so

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When $x > 0$, $\lim_{t \to 0} u(x,t) = 0 \Rightarrow \lim_{\zeta \uparrow +\infty} f(\zeta) = 0$

Proof of Lemma 3

$$\begin{cases} (f^m)'' + \frac{m-\beta}{2(1-\beta)}\zeta f' \frac{1}{1-\beta} f - bf^\beta = 0, \zeta \in \mathbb{R} \\ f(\zeta) \sim C(-\zeta)^{\frac{2}{m-\beta}} \quad \text{as } \zeta \downarrow -\infty, \; f(+\infty) = 0 \end{cases}$$

Lemma is proved.

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$$\text{If } C = C_* \equiv \left[\frac{b(m-\beta)^2}{2m(m+\beta)}\right]^{\frac{1}{m-\beta}} \text{ then } \zeta_* = 0$$

$f(\zeta) = C_*(-\zeta)_*^{\frac{2}{m-\beta}} \Rightarrow u(x,t) = C_*(-x)^{\frac{2}{m-\beta}}$ is a stationary solution

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$$u_t = (u^m)_{xx} - bu^\beta,\ x \in \mathbb{R},\ t > 0 \tag{20}$$

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Comparison Theorem implies

$$u_{-\epsilon} \le u \le u_\epsilon, \quad |x| \le |x_\epsilon|,\ 0 \le t \le \delta$$

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$$u_k(x,t) = u(x,t) = ku\big(k^{-1/\alpha}x, k^{\beta-1}t\big).$$

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Note: $\frac{2-\alpha(m-\beta)}{\alpha} < 0$.

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$$R = O(k^\theta) \text{ uniformly for } (x,t) \in E^k_{o\epsilon} \text{ as } k \to \infty,$$

$$\theta = \begin{cases} \frac{2-\alpha(m-\beta)}{\alpha}, & \text{if } \alpha < \frac{2}{m-1} \\ \beta - 1, & \text{if } \alpha \geq \frac{2}{m-1} \end{cases}$$

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$$\lim_{k' \to +\infty} u_{k'}^{\pm\epsilon}(x,t) = v_{\pm\epsilon}(x,t) \text{ which solves}$$

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$$v_t + bv^\beta = 0,\ x \in \mathbb{R},\ 0 < t \le t_0;\ v(x,0) = (C \pm \epsilon)(-x)_+^\alpha,\ x \in \mathbb{R}$$

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Let $\tau = k^{\beta-1}t$ so that $\tau \to 0$ as $k \to +\infty$; then

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$$u(x,t) \sim \left[C^{1-\beta}(-x)_+^{\alpha(1-\beta)} - b(1-\beta)t\right]^{\frac{1}{1-\beta}}$$

as $t \downarrow 0^+$ along $x = \eta_l(t)$. Lemma is proved.