

Evolution of Interfaces for the Reaction-Diffusion Equation

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FIT Colloquium

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Cauchy problem for the Reaction-Diffusion equation:

$$\mathcal{L}u \equiv u_t - (u^m)_{xx} + bu^\beta = 0, x \in \mathbb{R}, 0 < t < T, \quad (1)$$

$$u(x, 0) = u_0(x), x \in \mathbb{R} \quad (2)$$

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Barenblatt's problem: *Does interface expand, shrink or remain stationary? Find the short-time behavior of the interface function $\eta(t)$ and $u(x, t)$ near $x = \eta(t)$.*

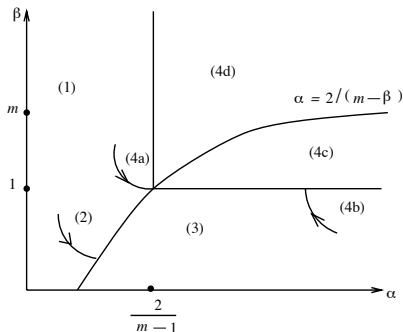
The **Answer** depends on m, β, b, C, α . Is it possible to give a full classification?

Full Classification

U. G. Abdulla, Reaction-diffusion in irregular domains, J. Differential Equations, 164, 2000, 321-354.

U. G. Abdulla and J. R. King, Interface development and local solutions to reaction-diffusion equations, SIAM J. Math. Anal., 32, 2, 2000, 235-260.

U. G. Abdulla, Evolution of interfaces and explicit asymptotics at infinity for the fast diffusion equation with absorption, Nonlinear Analysis, 50, 4, 2002, 541-560.



Lemma 1

Let u solve (1),(2),(4) with $m > 1, 0 < \alpha < \frac{2}{m-1}$. Then

$$u(x, t) = t^{\frac{\alpha}{2+\alpha(1-m)}} f(\xi), \xi = xt^{-\frac{1}{2+\alpha(1-m)}}, f(\xi) = u(\xi, 1)$$

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Let w be a solution of (1),(2),(4) with $C = 1$, and $f_0(\xi) = w(\xi, 1)$ be a corresponding solution of the nonlinear ODE. Then

$$f(\rho) = C^{\frac{2}{2+\alpha(1-m)}} f_0(C^{\frac{m-1}{\alpha(m-1)-2}} \rho), \xi_* = C^{\frac{m-1}{2-\alpha(m-1)}} \xi_*', \xi_*' = \sup\{\rho : f_0(\rho) > 0\}$$

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If u_0 satisfies (3) then $\eta(t) \sim \xi_* t^{\frac{1}{2+\alpha(1-m)}}$ as $t \downarrow 0$ and $\forall \rho < \xi_*$

$$u(x, t) \sim f(\rho) t^{\frac{\alpha}{2+\alpha(1-m)}} \quad \text{as } t \downarrow 0 \quad \text{along } x = \xi_\rho(t) = \rho t^{\frac{1}{2+\alpha(1-m)}}$$

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$$u(x, t) = kw(C^{\frac{1}{\alpha}} k^{-\frac{1}{\alpha}} x, C^{\frac{2}{\alpha}} k^{\frac{\alpha(m-1)-2}{\alpha}} t) \quad (6)$$

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$$\epsilon \downarrow 0 \Rightarrow u(\xi_{\rho}(t), t) \sim f(\rho; C) t^{\frac{\alpha}{2+\alpha(1-m)}} \quad \text{as } t \downarrow 0; \quad \forall \rho < \xi_*$$

$$\xi_*(C - \epsilon) t^{\frac{1}{2+\alpha(1-m)}} \leq \eta(t) \leq \xi_*(C + \epsilon) t^{\frac{1}{2+\alpha(1-m)}}, \quad 0 \leq t \leq \delta$$

Proof of Lemma 1: local case

$$f(\rho; C) = C^{\frac{2}{2+\alpha(1-m)}} f_0(C^{\frac{m-1}{\alpha(m-1)-2}} \rho)$$

Take in $u_{-\epsilon} \leq u \leq u_{\epsilon}$ $x = \xi_{\rho}(t)$, multiply by $t^{-\frac{\alpha}{2+\alpha(1-m)}} \Rightarrow$

$$u_{-\epsilon}(\xi_{\rho}(t), t) t^{-\frac{\alpha}{2+\alpha(1-m)}} \leq u(\xi_{\rho}(t), t) t^{-\frac{\alpha}{2+\alpha(1-m)}} \leq u_{\epsilon}(\xi_{\rho}(t), t) t^{-\frac{\alpha}{2+\alpha(1-m)}}$$

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$$\eta \sim \xi_* t^{\frac{1}{2+\alpha(1-m)}}, \quad \text{as } t \downarrow 0; \quad \xi_* = C^{\frac{m-1}{2+\alpha(1-m)}} \xi'_*$$

Lemma 1 is proved.

Summary of Lemma 1

Consider

$$\begin{cases} u_t = (u^m)_{xx}, & x \in \mathbb{R}, t > 0 \\ u(x, 0) = C(-x)_+^\alpha, & x \in \mathbb{R} \end{cases} \quad (7)$$

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$$\begin{cases} (f^m)'' + \xi f' - \frac{\alpha}{2-\alpha(m-1)} f = 0, & -\infty < \xi < \xi_* \\ f(-\infty) \sim C(-\xi)^\alpha, \quad f(\xi_*) = 0, \quad f(\xi) \equiv 0, & \xi \geq \xi_* \end{cases} \quad (8)$$

In fact, $f(\xi) = u(\xi, 1)$.

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$$f(\rho) = C^{\frac{2}{2-\alpha(m-1)}} f_0(C^{\frac{m-1}{2-\alpha(m-1)}} \rho)$$

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$C = 1$. If $u_0(x) \sim C(-x)_+^\alpha \Rightarrow \eta(t) \sim \xi_* t^{\frac{1}{2+\alpha(1-m)}}$ as $t \downarrow 0$ and $\forall \rho < \xi_*$

$$u(x, t) \sim f(\rho) t^{\frac{\alpha}{2+\alpha(1-m)}} \quad \text{as } t \downarrow 0 \quad \text{along } x = \xi_\rho(t) = \rho t^{\frac{1}{2+\alpha(1-m)}}$$

Lemma 2

Let u solve

$$u_t = (u^m)_{xx} - bu^\beta, \quad x \in \mathbb{R}, \quad t > 0 \quad (9)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R} \quad (10)$$

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a) $b > 0, 0 < \beta < 1 < m, 0 < \alpha < 2/(m - \beta)$

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$$u_0 \sim C(-x)_+^\alpha \Rightarrow \forall \epsilon > 0, \exists x_\epsilon < 0$$

$$(C - \epsilon/2)(-x)_+^\alpha \leq u_0(x) \leq (C + \epsilon/2)(-x)_+^\alpha, \quad x \geq x_\epsilon$$

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$$u_k^{\pm\epsilon}(x, t) = k u_{\pm\epsilon}(k^{-\frac{1}{\alpha}} x, k^{\frac{\alpha(m-1)-2}{\alpha}} t), \quad k \geq 0$$

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Calculate

$$\left(u_k^{\pm\epsilon}\right)_t - \left(u_k^{\pm\epsilon}\right)_{xx}^m = -b \left(u_k^{\pm\epsilon}\right)^\beta k^{\frac{\alpha(m-\beta)-2}{\alpha}}$$

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$$\begin{cases} u_t = (u^m)_{xx} - b k^{\frac{\alpha(m-\beta)-2}{\alpha}} u^\beta = 0, & x \in \mathbb{R}, t > 0 \\ u(x, 0) = (C \pm \epsilon)(-x)_+^\alpha, & x \in \mathbb{R} \end{cases}$$

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$$\alpha(m - \beta) - 2 < 0 \Rightarrow \lim_{k \rightarrow \infty} u_k^{\pm\epsilon}(x, t) = v_{\pm\epsilon}(x, t), \quad x \in \mathbb{R}, \quad t \geq 0 \quad (12)$$

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By the previous lemma 1

$$v_{\pm\epsilon}(\xi_\rho(t), t) = f(\rho, C \pm \epsilon)t^{\frac{\alpha}{2-\alpha(m-1)}}$$

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$$\lim_{k \rightarrow \infty} ku_{\pm\epsilon}(k^{-\frac{1}{\alpha}}\xi_\rho(t), k^{\frac{\alpha(m-1)-2}{\alpha}}t) = f(\rho; C \pm \epsilon)t^{\frac{\alpha}{2-\alpha(m-1)}}, \quad t \geq 0 \quad (13)$$

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$$\lim_{k \rightarrow \infty} k u_{\pm \epsilon}(k^{-\frac{1}{\alpha}} \xi_\rho(t), k^{\frac{\alpha(m-1)-2}{\alpha}} t) = f(\rho; C \pm \epsilon) t^{\frac{\alpha}{2 - \alpha(m-1)}}, \quad t \geq 0 \quad (13)$$

Let $k^{\frac{\alpha(m-1)-2}{\alpha}} t = \tau$ so $k^{-\frac{1}{\alpha}} \xi_\rho(t) = \xi_\rho(\tau)$ so from (13),

$$\lim_{\substack{k \rightarrow +\infty \\ (\tau \rightarrow 0)}} \frac{k u_{\pm \epsilon}(k^{-\frac{1}{\alpha}} \xi_\rho(t), k^{\frac{\alpha(m-1)-2}{\alpha}} t)}{t^{\frac{\alpha}{2 - \alpha(m-1)}}} = f(\rho; C \pm \epsilon)$$

Proof of Lemma 2

$$\lim_{\tau \rightarrow 0} \frac{u_{\pm\epsilon}(\xi_\rho(\tau), \tau)}{\tau^{\frac{\alpha}{2-\alpha(m-1)}}} = f(\rho; C \pm \epsilon)$$

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By comparison, from $u_{-\epsilon} \leq u \leq u_\epsilon$ it follows that

$$u(x, t) \sim f(\rho; C) t^{\frac{\alpha}{2-\alpha(m-1)}}, \quad t \downarrow 0 \tag{14}$$

Lemma 2 is proved except in case b) with $b < 0$.

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Lemma 2 is proved except in case b) with $b < 0$. Let $b < 0$, $\beta \geq 1$, $m > 1$, $0 < \alpha < 2/(m-1)$ and suppose $u_{\pm\epsilon}$ solves

$$\begin{cases} u_t - (u^m)_{xx} + bu^\beta = 0, & |x| < |x_\epsilon|, \quad 0 < t \leq \delta \\ u(x, 0) = (C \pm \epsilon)(-x)_+^\alpha \\ u(x_\epsilon, t) = (C \pm \epsilon)(-x_\epsilon)^\alpha, \quad u(-x_\epsilon, t) = 0, \quad 0 \leq t \leq \delta \end{cases} \tag{15}$$

Scale u as in the diffusion case

$$\begin{cases} u_k^{\pm\epsilon}(x, t) = k u_{\pm\epsilon}(k^{-\frac{1}{\alpha}}x, k^{\frac{\alpha(m-1)-2}{\alpha}}t), \quad k > 0 \\ u_t - (u^m)_{xx} + b k^{\frac{\alpha(m-\beta)-2}{\alpha}} u^\beta = 0, \quad D_\epsilon^k = \{|x| < k^{\frac{1}{\alpha}}|x_\epsilon|, \quad 0 < t \leq k^{\frac{2-\alpha(m-1)}{\alpha}}\delta\} \\ u(x, 0) = (C \pm \epsilon)(-x)_+^\alpha, \quad |x| < k^{\frac{1}{\alpha}}|x_\epsilon| \\ u(k^{\frac{1}{\alpha}}x_\epsilon, t) = k(C \pm \epsilon)(-x_\epsilon)^\alpha, \quad u(-k^{\frac{1}{\alpha}}x_\epsilon, t) = 0, \quad 0 \leq t \leq k^{\frac{2-\alpha(m-1)}{\alpha}}\delta \end{cases} \quad (16)$$

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$[UA, JDE] \Rightarrow \exists \delta > 0$ for which there is a local soln, and a comparison theorem implies

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Convergence of the $u_k^{\pm\epsilon}$: consider

$$g(x, t) = (C + 1)(1 + x^2)^{\alpha/2} (1 - \nu t)^{\frac{1}{1-m}}, \quad x \in \mathbb{R}, \quad 0 \leq t \leq t_0 = \frac{\nu^{-1}}{2}$$

Proof of Lemma 2

Calculate

$$L_k g \equiv g_t - (g^m)_{xx} + bk \frac{\alpha(m-\beta)-2}{\alpha} g^\beta = (C+1)(m-1)^{-1}(1+x^2)^{\frac{\alpha}{2}} (1-\nu t)^{\frac{m}{1-m}} S$$

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$$S = \nu - h(x) + b(m-1)(C+1)^{\beta-1} k \frac{\alpha(m-\beta)-2}{\alpha} (1+x^2)^{\frac{\alpha(\beta-1)}{2}} (1-\nu t)^{\frac{\beta-m}{1-m}}$$

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$$\nu = h_* + 1, \quad h_* = h(\alpha, m) = \max_{\mathbb{R}} h(x),$$

$$h(x) = (m-1)(C+1)^{m-1} \alpha m (1+x^2)^{\frac{\alpha(m-1)}{2}-2} (1+(\alpha m-1)x^2)$$

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Here $\frac{\alpha(m-1)-2}{2} < 0$ and $S \geq 1 + R$ where

$$R = O\left(k^{\frac{\alpha(m-\beta)-2}{\alpha}} \left(k^{\frac{2}{\alpha}}\right)^{\frac{\alpha(\beta-1)}{2}}\right) = O\left(k^{m-1-\frac{2}{\alpha}}\right)$$

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So when $k \rightarrow \infty$ $S > 0 \Rightarrow L_k g \geq 0$ on $D_{o\epsilon}^k = D_\epsilon^k \cap \{0 < t \leq t_0\}$

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$$L_k g \equiv g_t - (g^m)_{xx} + bk^{\frac{\alpha(m-\beta)-2}{\alpha}} g^\beta = (C+1)(m-1)^{-1}(1+x^2)^{\frac{\alpha}{2}} (1-\nu t)^{\frac{m}{1-m}} S$$

$$S = \nu - h(x) + b(m-1)(C+1)^{\beta-1} k^{\frac{\alpha(m-\beta)-2}{\alpha}} (1+x^2)^{\frac{\alpha(\beta-1)}{2}} (1-\nu t)^{\frac{\beta-m}{1-m}}$$

$$\nu = h_* + 1, \quad h_* = h(\alpha, m) = \max_{\mathbb{R}} h(x),$$

$$h(x) = (m-1)(C+1)^{m-1} \alpha m (1+x^2)^{\frac{\alpha(m-1)}{2}-2} (1+(\alpha m-1)x^2)$$

Here $\frac{\alpha(m-1)-2}{2} < 0$ and $S \geq 1 + R$ where

$$R = O\left(k^{\frac{\alpha(m-\beta)-2}{\alpha}} \left(k^{\frac{2}{\alpha}}\right)^{\frac{\alpha(\beta-1)}{2}}\right) = O\left(k^{m-1-\frac{2}{\alpha}}\right)$$

So when $k \rightarrow \infty$ $S > 0 \Rightarrow L_k g \geq 0$ on $D_{o\epsilon}^k = D_\epsilon^k \cap \{0 < t \leq t_0\}$

$$g(x, 0) \geq u_k^{\pm\epsilon}(x, 0)$$

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By comparison theorem,

$$0 \leq u_k^{\pm\epsilon}(x, t) \leq g(x, t) \text{ in } \overline{D_{o\epsilon}^k}$$

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For any compact $G \subset P = \{x \in \mathbb{R}, 0 < t \leq t_0\}$ $u_k^{\pm\epsilon}$ is uniformly bounded and $([UA, JDE])$ uniformly Hölder continuous $\Rightarrow \exists v_{\pm\epsilon}$ s.t.

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Since in

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$$\begin{aligned} u_t &= (u^m)_{xx}, \quad x \in \mathbb{R}, \quad 0 < t \leq t_0 \\ u(x, 0) &= (C \pm \epsilon)(-x)_+^\alpha, \quad x \in \mathbb{R} \end{aligned}$$

The rest of the proof is similar to the previous case.

Proof of Lemma 2

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Let $k^{\frac{\alpha(m-1)-2}{\alpha}}t = \tau$ so $k^{-\frac{1}{\alpha}}\xi_\rho(t) = \xi_\rho(\tau)$ so from (17),

$$\lim_{\substack{k \rightarrow +\infty \\ (\tau \rightarrow 0)}} \frac{ku_{\pm\epsilon}(k^{-\frac{1}{\alpha}}\xi_\rho(t), k^{\frac{\alpha(m-1)-2}{\alpha}}t)}{t^{\frac{\alpha}{2-\alpha(m-1)}}} = f(\rho; C \pm \epsilon)$$

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$$u_{\pm\epsilon}(\xi_\rho(\tau), \tau) \sim f(\rho; C \pm \epsilon)\tau^{\frac{\alpha}{2-\alpha(m-1)}}, \quad \tau \downarrow 0$$

By comparison, from $u_{-\epsilon} \leq u \leq u_\epsilon$ it follows that

$$u(x, t) \sim f(\rho; C)t^{\frac{\alpha}{2-\alpha(m-1)}}, \quad t \downarrow 0 \quad \text{along } x = \xi_\rho(t) = \rho t^{\frac{1}{2+\alpha(1-m)}}$$

Lemma is proved.

Lemma 3

Let u solve

$$u_t = (u^m)_{xx} - bu^\beta, \quad x \in \mathbb{R}, \quad t > 0 \quad (18)$$

$$u(x, 0) = C(-x)_+^\alpha, \quad x \in \mathbb{R}, \quad \alpha = 2/(m - \beta) \quad (19)$$

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Where $b > 0$, $0 < \beta < 1$, $m \geq 1$. Then

$$u(x, t) = t^{\frac{1}{1-\beta}} f_1(\zeta), \quad \zeta = \frac{x}{t^{\frac{m-\beta}{2(1-\beta)}}}$$

Proof of Lemma 3

Choose $v(x, t) = ku(k^{\gamma_1}x, k^{\gamma_2}t)$ so

$$v_t - (v^m)_{xx} + bv^\beta = k^{1+\gamma_2}u_t - k^{m+2\gamma_1}(u^m)_{xx} + bk^\beta u^\beta$$

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Hence $f(\zeta) = u(\zeta, 1)$.

Proof of Lemma 3

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$$\begin{aligned} u_t - (u^m)_{xx} + bu^\beta &= \frac{1}{1-\beta} t^{\frac{\beta}{1-\beta}} f - \frac{m-\beta}{2(1-\beta)} \zeta f' t^{\frac{\beta}{1-\beta}} - \\ &\quad - t^{\frac{\beta}{1-\beta}} (f^m)'' + bt^{\frac{\beta}{1-\beta}} f^\beta = 0 \end{aligned}$$

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When $x < 0$, $\lim_{t \rightarrow 0} u(x, t) = C(-x)_+^{\frac{2}{m-\beta}}$, so

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When $x > 0$, $\lim_{t \rightarrow 0} u(x, t) = 0 \Rightarrow \lim_{\zeta \uparrow +\infty} f(\zeta) = 0$

Proof of Lemma 3

$$\begin{cases} (f^m)'' + \frac{m-\beta}{2(1-\beta)} \zeta f' \frac{1}{1-\beta} f - b f^\beta = 0, \zeta \in \mathbb{R} \\ f(\zeta) \sim C(-\zeta)^{\frac{2}{m-\beta}} \quad \text{as } \zeta \downarrow -\infty, \quad f(+\infty) = 0 \end{cases}$$

Lemma is proved.

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$$\text{If } C = C_* \equiv \left[\frac{b(m-\beta)^2}{2m(m+\beta)} \right]^{\frac{1}{m-\beta}} \text{ then } \zeta_* = 0$$

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Let u solve

$$u_t = (u^m)_{xx} - bu^\beta, \quad x \in \mathbb{R}, \quad t > 0 \quad (20)$$

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Comparison Theorem implies

$$u_{-\epsilon} \leq u \leq u_\epsilon, \quad |x| \leq |x_\epsilon|, \quad 0 \leq t \leq \delta$$

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$$u_k(x, t) = u(x, t) = ku(k^{-1/\alpha}x, k^{\beta-1}t).$$

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$R = O(k^\theta)$ uniformly for $(x, t) \in E_{o\epsilon}^k$ as $k \rightarrow \infty$,

$$\theta = \begin{cases} \frac{2-\alpha(m-\beta)}{\alpha}, & \text{if } \alpha < \frac{2}{m-1} \\ \beta - 1, & \text{if } \alpha \geq \frac{2}{m-1} \end{cases}$$

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$$\lim_{k' \rightarrow +\infty} u_{k'}^{\pm\epsilon}(x, t) = v_{\pm\epsilon}(x, t) \text{ which solves}$$

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$$u(x, t) \sim \left[C^{1-\beta} (-x)_+^{\alpha(1-\beta)} - b(1-\beta)t \right]^{\frac{1}{1-\beta}}$$

as $t \downarrow 0^+$ along $x = \eta_l(t)$. Lemma is proved.