Evolution of Interfaces for the Reaction-Diffusion Equation

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Introduction

Cauchy problem for the Reaction-Diffusion equation:

$$\mathcal{L}u \equiv u_t - (u^m)_{xx} + bu^\beta = 0, x \in \mathbb{R}, 0 < t < T, \tag{1}$$

$$u(x,0) = u_0(x), x \in \mathbb{R} \tag{2}$$

where $m > 1, b \in \mathbb{R}, \beta > 0, 0 < T \leq +\infty$, $u_0 \geq 0$, $u_0 \in C(\mathbb{R})$.

$$\eta(t) = \sup\{x : u(x,t) > 0\}, \quad \eta(0) = 0$$

$$u_0(x) \sim C(-x)_+^\alpha, \quad \text{as } x \to 0-, \quad \text{for some } C > 0, \alpha > 0. \tag{3}$$

$$u_0(x) = C(-x)_+^\alpha, x \in \mathbb{R} \tag{4}$$

Barenblatt's problem: *Does interface expand, shrink or remain stationary? Find the short-time behavior of the interface function $\eta(t)$ and $u(x,t)$ near $x = \eta(t)$.*

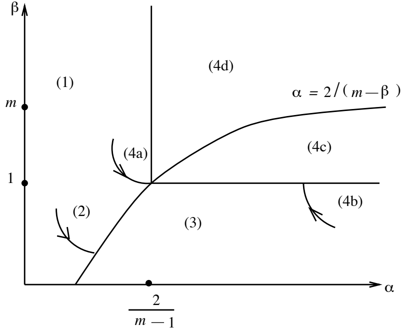
The **Answer** depends on m, β, b, C, α. Is it possible to give a full classification?

Full Classification

U. G. Abdulla, Reaction-diffusion in irregular domains, J. Differential Equations, 164, 2000, 321-354.

U. G. Abdulla and J. R. King, Interface development and local solutions to reaction-diffusion equations, SIAM J. Math. Anal., 32, 2, 2000, 235-260.

U. G. Abdulla, Evolution of interfaces and explicit asymptotics at infinity for the fast diffusion equation with absorption, Nonlinear Analysis, 50, 4, 2002, 541-560.



Lemma 1

Let u *solve* (1), (2), (4) *with* $m > 1, 0 < \alpha < \frac{2}{m-1}, b = 0$. *Then*

$$u(x,t) = t^{\frac{\alpha}{2+\alpha(1-m)}} f(\xi), \xi = xt^{-\frac{1}{2+\alpha(1-m)}},\ f(\xi) = u(\xi, 1)$$

$$\frac{d^2 f^m}{d\xi^2} + \frac{1}{2+\alpha(1-m)} \xi \frac{df}{d\xi} - \frac{\alpha}{2+\alpha(1-m)} f = 0, \xi \in \mathbb{R},$$

$$f \sim C(-\xi)^{\alpha} \text{as } \xi \downarrow -\infty, f(+\infty) = 0$$

$\exists \xi_*$ *such that* $f(\xi) > 0, -\infty < \xi < \xi_*; f(\xi) \equiv 0, \xi \geq \xi_*,\ \eta(t) = \xi_* t^{\frac{1}{2+\alpha(1-m)}}$

Let w *be a solution of* (1), (2), (4) *with* $C = 1$ *(*$b = 0$*), and* $f_0(\xi) = w(\xi, 1)$ *be a corresponding solution of the nonlinear ODE. Then*

$$f(\rho) = C^{\frac{2}{2+\alpha(1-m))}} f_0(C^{\frac{m-1}{\alpha(m-1)-2}} \rho), \xi_* = C^{\frac{m-1}{2-\alpha(m-1)}} \xi'_*, \xi'_* = \sup\{\rho : f_0(\rho) > 0\}$$

If u_0 *satisfies (3) then* $\eta(t) \sim \xi_* t^{\frac{1}{2+\alpha(1-m)}}$ *as* $t \downarrow 0$ *and* $\forall \rho < \xi_*$

$$u(x,t) \sim f(\rho) t^{\frac{\alpha}{2+\alpha(1-m)}} \quad \text{as } t \downarrow 0 \quad \text{along } x = \xi_\rho(t) = \rho t^{\frac{1}{2+\alpha(1-m)}}$$

Proof of Lemma 1: global case

In this proof, assume $b = 0$. Consider

$$v(x,t) = ku(k^{\gamma}x, k^{\mu}t)$$

$\gamma = -\frac{1}{\alpha}, \mu = \frac{\alpha(m-1)-2}{\alpha} \Rightarrow v$ solves (1), (2), (4) $\Rightarrow$ due to uniqueness.

$$v(x,t) = u(x,t) = ku(k^{-\frac{1}{\alpha}}x, k^{\frac{\alpha(m-1)-2}{\alpha}}t) \tag{5}$$

Choose $k = t^{\frac{\alpha}{2-\alpha(m-1)}} \Rightarrow u(x,t) = t^{\frac{\alpha}{2+\alpha(1-m)}}u(xt^{-\frac{1}{2+\alpha(1-m)}}, 1)$
Find dependence of u (or f) on C: $v = C^{-1}u$ satisfies:

$$C^{1-m}v_t = (v^m)_{xx}, x \in \mathbb{R}, 0 < t < T;\ v(x,0) = (-x)_+^{\alpha}, x \in \mathbb{R}$$

Choose $\tau = C^{m-1}t, w(x,\tau) = v(x, C^{1-m}\tau) \Rightarrow w$ solves (1), (2), (4) with $C = 1$. Hence $u(x,t) = Cw(x, C^{m-1}t)$. From (5) $\Rightarrow$

$$u(x,t) = kw(C^{\frac{1}{\alpha}}k^{-\frac{1}{\alpha}}x, C^{\frac{2}{\alpha}}k^{\frac{\alpha(m-1)-2}{\alpha}}t) \tag{6}$$

$k = (C^{\frac{2}{\alpha}}t)^{\frac{\alpha}{2-\alpha(m-1)}} \Rightarrow u(x,t) = C^{\frac{2}{2-\alpha(m-1)}}w(C^{\frac{m-1}{\alpha(m-1)-2}}\xi, 1)t^{\frac{\alpha}{2-\alpha(m-1)}}$

Proof of Lemma 1: local case

$$u(x,t) = t^{\frac{\alpha}{2+\alpha(1-m)}} f(\xi) \Rightarrow f(\rho) = C^{\frac{2}{2+\alpha(1-m))}} f_0(C^{\frac{m-1}{\alpha(m-1)-2}}\rho)$$

$$\Rightarrow \xi_* = C^{\frac{m-1}{2-\alpha(m-1)}} \xi_*' \Rightarrow \quad \text{global case is proved}$$

Local case: $u_0(x) \sim C(-x)_+^{\alpha}, \quad$ as $x \to 0-, \quad$ for some $C > 0, \alpha > 0$

$$\forall \epsilon > 0,\ \exists x_\epsilon < 0,\ \left(C - \frac{\epsilon}{2}\right)(-x)_+^{\alpha} \le u_0(x) \le \left(C + \frac{\epsilon}{2}\right)(-x)_+^{\alpha},\ x \ge x_\epsilon$$

Let $u_{\pm\epsilon}(x,t)$ be a solution of (1), (2) with $u_0(x) = (C \pm \epsilon)(-x)_+^{\alpha}$

$$\exists \delta = \delta(\epsilon) > 0 \ \ u_\epsilon(x_\epsilon, t) \ge u(x_\epsilon, t),\ u_{-\epsilon}(x_\epsilon, t) \le u(x_\epsilon, t),\ 0 \le t \le \delta$$

Comparison Theorem $\Rightarrow u_{-\epsilon} \le u \le u_\epsilon,\ x \ge x_\epsilon, 0 \le t \le \delta$

$$u_{\pm\epsilon}(x,t) = f(\xi; C \pm \epsilon) t^{\frac{\alpha}{2+\alpha(1-m)}}, \xi = xt^{-\frac{1}{2+\alpha(1-m)}}$$

$$u_{\pm\epsilon}(\xi_\rho(t), t) = f(\rho; C \pm \epsilon) t^{\frac{\alpha}{2+\alpha(1-m)}}, \xi_\rho(t) = \rho t^{\frac{1}{2+\alpha(1-m)}},\ \rho < \xi_*$$

Proof of Lemma 1: local case

$$f(\rho; C) = C^{\frac{2}{2+\alpha(1-m))}} f_0(C^{\frac{m-1}{\alpha(m-1)-2}} \rho)$$

Take in $u_{-\epsilon} \leq u \leq u_\epsilon$ $x = \xi_\rho(t)$, multiply by $t^{-\frac{\alpha}{2+\alpha(1-m)}} \Rightarrow$

$$u_{-\epsilon}(\xi_\rho(t), t) t^{-\frac{\alpha}{2+\alpha(1-m}} \leq u(\xi_\rho(t), t) t^{-\frac{\alpha}{2+\alpha(1-m)}} \leq u_\epsilon(\xi_\rho(t), t) t^{-\frac{\alpha}{2+\alpha(1-m)}}$$

$$f(\rho; C - \epsilon) \leq u(\xi_\rho(t), t) t^{-\frac{\alpha}{2+\alpha(1-m)}} \leq f(\rho; C + \epsilon), \ 0 \leq t \leq \delta$$

$$f(\rho; C - \epsilon) \leq \liminf_{t \downarrow 0} u(\xi_\rho(t), t) t^{\frac{\alpha}{\alpha(m-1)-2}} \leq \limsup_{t \downarrow 0} \cdots \leq f(\rho; C + \epsilon)$$

$$\epsilon \downarrow 0 \ \Rightarrow u(\xi_\rho(t), t) \sim f(\rho; C) t^{\frac{\alpha}{2+\alpha(1-m)}} \quad \text{as } t \downarrow 0; \ \forall \rho < \xi_*$$

$$\xi_*(C - \epsilon) t^{\frac{1}{2+\alpha(1-m)}} \leq \eta(t) \leq \xi_*(C + \epsilon) t^{\frac{1}{2+\alpha(1-m)}}, \ 0 \leq t \leq \delta$$

$$\eta \sim \xi_* t^{\frac{1}{2+\alpha(1-m)}}, \quad \text{as } t \downarrow 0; \ \xi_* = C^{\frac{m-1}{2+\alpha(1-m)}} \xi'_*$$

Lemma 1 is proved.

Summary of Lemma 1

Consider

$$\begin{cases} u_t = (u^m)_{xx}, \ x \in \mathbb{R}, \ t > 0 \\ u(x,0) = C(-x)_+^{\alpha}, \ x \in \mathbb{R} \end{cases} \tag{7}$$

$$u(x,t) = t^{\frac{\alpha}{2-\alpha(m-1)}} f(\xi), \ \xi = \frac{x}{t^{\frac{1}{2-\alpha(m-1)}}}$$

$$\begin{cases} (f^m)'' + \xi f' - \frac{\alpha}{2-\alpha(m-1)} f = 0, \ \ -\infty < \xi < \xi_* \\ f(-\infty) \sim C(-\xi)^{\alpha}, \ f(\xi_*) = 0, \ f(\xi) \equiv 0, \ \xi \geq \xi_* \end{cases} \tag{8}$$

In fact, $f(\xi) = u(\xi, 1)$. Scaling out $C \to 1$:

$$f(\rho) = C^{\frac{2}{2-\alpha(m-1)}} f_0(C^{\frac{m-1}{2-\alpha(m-1)}} \rho)$$

f_0 solves (8) with $C = 1$; $f_0(\xi) = w(\xi, 1)$, $w(\xi, 1)$ solves (7) with $C = 1$.
If $u_0(x) \sim C(-x)_+^{\alpha} \Rightarrow \eta(t) \sim \xi_* t^{\frac{1}{2+\alpha(1-m)}}$ as $t \downarrow 0$ and $\forall \rho < \xi_*$

$$u(x,t) \sim f(\rho) t^{\frac{\alpha}{2+\alpha(1-m)}} \quad \text{as } t \downarrow 0 \quad \text{along } x = \xi_\rho(t) = \rho t^{\frac{1}{2+\alpha(1-m)}}$$

Lemma 2

Lemma 2

Let u *solve*

$$u_t = (u^m)_{xx} - bu^\beta,\ x \in \mathbb{R},\ t > 0 \tag{9}$$

$$u(x,0) = u_0(x),\ x \in \mathbb{R} \tag{10}$$

$$u_0 \sim C(-x)_+^\alpha,\ \text{as } x \uparrow 0 \tag{11}$$

a) $b > 0$, $0 < \beta < 1 < m$, $0 < \alpha < 2/(m-\beta)$

b) $b \neq 0$, $\beta \geq 1$, $m > 1$, $0 < \alpha < 2/(m-1)$

Then

$$u(x,t) \sim f(\rho) t^{\frac{\alpha}{2-\alpha(m-1)}}, \quad x = \xi_\rho(t) = \rho t^{\frac{1}{2-\alpha(m-1)}}$$

Proof of Lemma 2

$u_0 \sim C(-x)_+^\alpha \Rightarrow \forall \epsilon > 0,\ \exists x_\epsilon < 0$

$$(C-\epsilon/2)(-x)_+^\alpha \le u_0(x) \le (C+\epsilon/2)(-x)_+^\alpha,\ x \ge x_\epsilon$$

$$\exists \delta(\epsilon) > 0 :\ u_{-\epsilon}(x_\epsilon, t) \le u(x_\epsilon, t) \le u_\epsilon(x_\epsilon, t),\ 0 \le t \le \delta$$

$$u_{-\epsilon} \le u \le u_\epsilon,\ x \ge x_\epsilon,\ 0 \le t \le \delta$$

Scale u as in the diffusion case

$$u_k^{\pm\epsilon}(x,t) = k u_{\pm\epsilon}(k^{-\frac{1}{\alpha}} x, k^{\frac{\alpha(m-1)-2}{\alpha}} t),\ k \ge 0$$

Calculate

$$\big(u_k^{\pm\epsilon}\big)_t - \big(u_k^{\pm\epsilon}\big)^m_{xx} = -b\big(u_k^{\pm\epsilon}\big)^\beta k^{\frac{\alpha(m-\beta)-2}{\alpha}}$$

$$\begin{cases} u_t = (u^m)_{xx} - bk^{\frac{\alpha(m-\beta)-2}{\alpha}} u^\beta, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = (C \pm \epsilon)(-x)_+^\alpha,\ x \in \mathbb{R} \end{cases}$$

Proof of Lemma 2

$$\alpha(m-\beta)-2<0 \;\Rightarrow\; \lim_{k\to\infty} u_k^{\pm\epsilon}(x,t)=v_{\pm\epsilon}(x,t),\; x\in\mathbb{R},\; t\geq 0 \quad (12)$$

$$v_{\pm\epsilon} \text{ solves } \begin{cases} v_t=(v^m)_{xx} \\ v\Big|_{t=0}=(C\pm\epsilon)(-x)_+^{\alpha},\; T=+\infty \end{cases}$$

By the previous lemma 1

$$v_{\pm\epsilon}(\xi_\rho(t),t)=f(\rho,C\pm\epsilon)t^{\frac{\alpha}{2-\alpha(m-1)}}$$

From (12)

$$\lim_{k\to\infty} ku_{\pm\epsilon}(k^{-\frac{1}{\alpha}}\xi_\rho(t),k^{\frac{\alpha(m-1)-2}{\alpha}}t)=f(\rho;C\pm\epsilon)t^{\frac{\alpha}{2-\alpha(m-1)}},\; t\geq 0 \quad (13)$$

Let $k^{\frac{\alpha(m-1)-2}{\alpha}}t=\tau$ so $k^{-\frac{1}{\alpha}}\xi_\rho(t)=\xi_\rho(\tau)$ so from (13),

$$\lim_{\substack{k\to+\infty \\ (\tau\to 0)}} \frac{ku_{\pm\epsilon}(k^{-\frac{1}{\alpha}}\xi_\rho(t),k^{\frac{\alpha(m-1)-2}{\alpha}}t)}{t^{\frac{\alpha}{2-\alpha(m-1)}}}=f(\rho;C\pm\epsilon)$$

Proof of Lemma 2

$$\lim_{\tau \to 0} \frac{u_{\pm\epsilon}(\xi_\rho(\tau), \tau)}{\tau^{\frac{\alpha}{2-\alpha(m-1)}}} = f(\rho; C \pm \epsilon)$$

So

$$u_{\pm\epsilon}(\xi_\rho(\tau), \tau) \sim f(\rho; C \pm \epsilon)\tau^{\frac{\alpha}{2-\alpha(m-1)}}, \ \tau \downarrow 0$$

By comparison, from $u_{-\epsilon} \leq u \leq u_\epsilon$ it follows that

$$u(x,t) \sim f(\rho; C)t^{\frac{\alpha}{2-\alpha(m-1)}}, \ t \downarrow 0 \tag{14}$$

Lemma 2 is proved except in case b) with $b < 0$. Let $b < 0$, $\beta \geq 1$, $m > 1$, $0 < \alpha < 2/(m-1)$ and suppose $u_{\pm\epsilon}$ solves

$$\begin{cases} u_t - (u^m)_{xx} + bu^\beta = 0, \ |x| < |x_\epsilon|, \ 0 < t \leq \delta \\ u(x,0) = (C \pm \epsilon)(-x)_+^\alpha \\ u(x_\epsilon, t) = (C \pm \epsilon)(-x_\epsilon)^\alpha, \ u(-x_\epsilon, t) = 0, \ 0 \leq t \leq \delta \end{cases} \tag{15}$$

Proof of Lemma 2

Scale u as in the diffusion case

$$\begin{cases} u_k^{\pm\epsilon}(x,t) = ku_{\pm\epsilon}(k^{-\frac{1}{\alpha}}x, k^{\frac{\alpha(m-1)-2}{\alpha}}t), \ k>0 \\ u_t - (u^m)_{xx} + bk^{\frac{\alpha(m-\beta)-2}{\alpha}}u^\beta = 0, \ D_\epsilon^k = \{|x| < k^{\frac{1}{\alpha}}|x_\epsilon|, \ 0 < t \le k^{\frac{2-\alpha(m-1)}{\alpha}}\delta\} \\ u(x,0) = (C\pm\epsilon)(-x)_+^\alpha, \ |x| < k^{\frac{1}{\alpha}}|x_\epsilon| \\ u(k^{\frac{1}{\alpha}}x_\epsilon, t) = k(C\pm\epsilon)(-x_\epsilon)^\alpha, \ u(-k^{\frac{1}{\alpha}}x_\epsilon, t) = 0, \ 0 \le t \le k^{\frac{2-\alpha(m-1)}{\alpha}}\delta \end{cases} \tag{16}$$

$[UA, JDE] \Rightarrow \exists\delta > 0$ for which there is a local soln, and a comparison theorem implies

$$u_{-\epsilon} \le u \le u_\epsilon, \ |x| < |x_\epsilon|, \ 0 \le t \le \delta$$

Convergence of the $u_k^{\pm\epsilon}$: consider

$$g(x,t) = (C+1)(1+x^2)^{\alpha/2}(1-\nu t)^{\frac{1}{1-m}}, \ x\in\mathbb{R}, \ 0 \le t \le t_0 = \frac{\nu^{-1}}{2}$$

Proof of Lemma 2

Calculate

$$L_k g \equiv g_t - (g^m)_{xx} + bk^{\frac{\alpha(m-\beta)-2}{\alpha}} g^\beta = (C+1)(m-1)^{-1}(1+x^2)^{\frac{\alpha}{2}}(1-\nu t)^{\frac{m}{1-m}} S$$

$$S = \nu - h(x) + b(m-1)(C+1)^{\beta-1} k^{\frac{\alpha(m-\beta)-2}{\alpha}} (1+x^2)^{\frac{\alpha(\beta-1)}{2}} (1-\nu t)^{\frac{\beta-m}{1-m}}$$

$$\nu = h_* + 1, \; h_* = h(\alpha, m) = \max_{\mathbb{R}} h(x),$$

$$h(x) = (m-1)(C+1)^{m-1}\alpha m(1+x^2)^{\frac{\alpha(m-1)}{2}-2}\Big(1+(\alpha m-1)x^2\Big)$$

Here $\frac{\alpha(m-1)-2}{2} < 0$ and $S \geq 1 + R$ where

$$R = O\Big(k^{\frac{\alpha(m-\beta)-2}{\alpha}} \big(k^{\frac{2}{\alpha}}\big)^{\frac{\alpha(\beta-1)}{2}}\Big) = O\Big(k^{m-1-\frac{2}{\alpha}}\Big)$$

So when $k \to \infty$ $S > 0 \Rightarrow L_k g \geq 0$ on $D^k_{o\epsilon} = D^k_\epsilon \cap \{0 < t \leq t_0\}$

$$g(x,0) \geq u_k^{\pm\epsilon}(x,0)$$

$$g(\pm k^{\frac{1}{\alpha}} x_\epsilon, t) \sim (C+1)(1+k^{\frac{2}{\alpha}} x_\epsilon^2)^{\frac{\alpha}{2}} \geq (C+1)k|x_\epsilon|^\alpha \geq u_k^{\pm\epsilon}(\pm k^{\frac{1}{\alpha}} x_\epsilon, t), \; 0 \leq t \leq t_0$$

Proof of Lemma 2

By comparison theorem,

$$0 \leq u_k^{\pm\epsilon}(x,t) \leq g(x,t) \text{ in } \overline{D_{o\epsilon}^k}$$

For any compact $G \subset P = \{x \in \mathbb{R}, 0 < t \leq t_0\}$ $u_k^{\pm\epsilon}$ is uniformly bounded and ($[UA, JDE]$) uniformly Hölder continuous $\Rightarrow \exists v_{\pm\epsilon}$ s.t.

$$\lim_{k' \to +\infty} u_{k'}^{\pm\epsilon} = v_{\pm\epsilon}(x,t), \; (x,t) \in P$$

Since in

$$u_t - (u^m)_{xx} + bk^{\frac{\alpha(m-\beta)-2}{\alpha}} u^\beta = 0$$

the exponent $\alpha(m-\beta) - 2 < 0$, it follows that $v_{\pm\epsilon}$ solves

$$u_t = (u^m)_{xx}, \; x \in \mathbb{R}, \; 0 < t \leq t_0$$
$$u(x,0) = (C \pm \epsilon)(-x)_+^\alpha, \; x \in \mathbb{R}$$

The rest of the proof is similar to the previous case.

Proof of Lemma 2

By the previous lemma 1

$$v_{\pm\epsilon}(\xi_\rho(t),t) = f(\rho, C\pm\epsilon)t^{\frac{\alpha}{2-\alpha(m-1)}}$$

$$\lim_{k\to\infty} ku_{\pm\epsilon}(k^{-\frac{1}{\alpha}}\xi_\rho(t), k^{\frac{\alpha(m-1)-2}{\alpha}}t) = f(\rho; C\pm\epsilon)t^{\frac{\alpha}{2-\alpha(m-1)}},\ t\geq 0 \quad (17)$$

Let $k^{\frac{\alpha(m-1)-2}{\alpha}}t = \tau$ so $k^{-\frac{1}{\alpha}}\xi_\rho(t) = \xi_\rho(\tau)$ so from (17),

$$\lim_{\substack{k\to+\infty \\ (\tau\to 0)}} \frac{ku_{\pm\epsilon}(k^{-\frac{1}{\alpha}}\xi_\rho(t), k^{\frac{\alpha(m-1)-2}{\alpha}}t)}{t^{\frac{\alpha}{2-\alpha(m-1)}}} = f(\rho; C\pm\epsilon)$$

$$\lim_{\tau\to 0} \frac{u_{\pm\epsilon}(\xi_\rho(\tau),\tau)}{\tau^{\frac{\alpha}{2-\alpha(m-1)}}} = f(\rho; C\pm\epsilon)$$

$$u_{\pm\epsilon}(\xi_\rho(\tau),\tau) \sim f(\rho; C\pm\epsilon)\tau^{\frac{\alpha}{2-\alpha(m-1)}},\ \tau\downarrow 0$$

By comparison, from $u_{-\epsilon}\leq u\leq u_\epsilon$ it follows that

$$u(x,t)\sim f(\rho;C)t^{\frac{\alpha}{2-\alpha(m-1)}},\ t\downarrow 0 \quad \text{along } x=\xi_\rho(t)=\rho t^{\frac{1}{2+\alpha(1-m)}}$$

Lemma is proved.

Lemma 3

Let u solve

$$u_t = (u^m)_{xx} - bu^\beta,\ x \in \mathbb{R},\ t > 0 \tag{18}$$

$$u(x,0) = C(-x)_+^\alpha,\ x \in \mathbb{R},\ \alpha = 2/(m-\beta) \tag{19}$$

Where $b > 0$, $0 < \beta < 1$, $m \geq 1$. Then

$$u(x,t) = t^{\frac{1}{1-\beta}} f_1(\zeta),\ \zeta = \frac{x}{t^{\frac{m-\beta}{2(1-\beta)}}}$$

Proof of Lemma 3

Choose $v(x,t) = ku(k^{\gamma_1}x, k^{\gamma_2}t)$ so

$$v_t - (v^m)_{xx} + bv^\beta = k^{1+\gamma_2}u_t - k^{m+2\gamma_1}(u^m)_{xx} + bk^\beta u^\beta$$

Choose $1 + \gamma_2 = m + 2\gamma_1 = \beta$.

$$v(x,0) = u(k^{\gamma_1}x, 0) = kC(-x)_+^\alpha k^{\gamma_1\alpha} = k^{1+\gamma_1\alpha}C(-x)_+^\alpha$$

Choose $1 + \gamma_1\alpha = 1 + \gamma_1\frac{2}{m-\beta} = 0$ so $\gamma_1 = (\beta - m)/2$ and $\gamma_2 = \beta - 1$. Hence

$$v(x,t) = ku(k^{\frac{\beta-m}{2}}x, k^{\beta-1}t) \text{ solves (18), (19)}$$

$$\text{Uniqueness} \Rightarrow u(x,t) = ku(k^{\frac{\beta-m}{2}}x, k^{\beta-1}t)$$

Choose $k^{\beta-1}t = 1$,

$$u(x,t) = t^{\frac{1}{1-\beta}}u\left(\frac{x}{t^{\frac{m-\beta}{2(1-\beta)}}}, 1\right) = t^{\frac{1}{1-\beta}}u(\zeta, 1)$$

Hence $f(\zeta) = u(\zeta, 1)$.

Proof of Lemma 3

with

$$u(x,t) = t^{\frac{1}{1-\beta}} f(\zeta), \quad \zeta = \frac{x}{t^{\frac{m-\beta}{2(1-\beta)}}}$$

Calculate

$$u_t - (u^m)_{xx} + bu^\beta = \frac{1}{1-\beta} t^{\frac{\beta}{1-\beta}} f - \frac{m-\beta}{2(1-\beta)} \zeta f' t^{\frac{\beta}{1-\beta}} -$$
$$- t^{\frac{\beta}{1-\beta}} (f^m)'' + bt^{\frac{\beta}{1-\beta}} f^\beta = 0$$

When $x < 0$, $\lim_{t\to 0} u(x,t) = C(-x)_+^{\frac{2}{m-\beta}}$, so

$$u(x,t) = t^{\frac{1}{1-\beta}} f\left(\frac{x}{t^{\frac{m-\beta}{2(1-\beta)}}}\right) \Rightarrow \lim_{t\downarrow 0} \frac{u(x,t)}{C(-x)^{\frac{2}{m-\beta}}} =$$
$$= \lim_{t\downarrow 0} \frac{t^{\frac{1}{1-\beta}} f\left(\frac{x}{t^{\frac{m-\beta}{2(1-\beta)}}}\right)}{C(-x)^{\frac{2}{m-\beta}}} = \lim_{\zeta\downarrow -\infty} \frac{f(\zeta)}{C(-\zeta)^{\frac{2}{m-\beta}}} = 1$$

When $x > 0$, $\lim_{t\to 0} u(x,t) = 0 \Rightarrow \lim_{\zeta\uparrow +\infty} f(\zeta) = 0$

Proof of Lemma 3

$$\begin{cases} (f^m)'' + \frac{m-\beta}{2(1-\beta)}\zeta f' \frac{1}{1-\beta} f - bf^\beta = 0, \zeta \in \mathbb{R} \\ f(\zeta) \sim C(-\zeta)^{\frac{2}{m-\beta}} \quad \text{as } \zeta \downarrow -\infty, \; f(+\infty) = 0 \end{cases}$$

Lemma is proved.
$\exists \zeta_*$ st. $f \equiv 0$ if $\zeta \geq \zeta_*$; $f(\zeta) > 0$, if $\zeta < \zeta_*$.

$$\text{If } C = C_* \equiv \left[\frac{b(m-\beta)^2}{2m(m+\beta)} \right]^{\frac{1}{m-\beta}} \text{ then } \zeta_* = 0$$

$f(\zeta) = C_*(-\zeta)_*^{\frac{2}{m-\beta}} \Rightarrow u(x,t) = C_*(-x)^{\frac{2}{m-\beta}}$ is a stationary solution
If $C \lesseqgtr C_*$ then $\zeta_* \lesseqgtr 0$; If $C > C_* \Rightarrow \zeta_* > 0 \Rightarrow f(0) = A$ where $A(m,\beta,C,b) > 0 \Rightarrow u(0,t) = At^{\frac{1}{1-\beta}}$
If $u_0(x) \sim C(-x)_+^{\frac{2}{m-\beta}}$ as $x \uparrow 0$ and $C > C_* \Rightarrow u(0,t) \sim At^{\frac{1}{1-\beta}}$ as $t \downarrow 0$.

Lemma 4

Let u solve

$$u_t = (u^m)_{xx} - bu^\beta, \ x \in \mathbb{R}, \ t > 0 \tag{20}$$

$$u(x,0) = u_0(x), \ x \in \mathbb{R} \tag{21}$$

$$u_0 \sim C(-x)_+^\alpha, \ \text{as } x \uparrow 0 \tag{22}$$

If $b > 0$, $0 < \beta < 1$, $\alpha > \frac{2}{m-\beta}$, *then* $\forall l > l_* = C^{-1/\alpha}\big(b(1-\beta)\big)^{\frac{1}{1-\beta}}$

$$u(x,t) \sim \Big[C^{1-\beta}(-x)_+^{\alpha(1-\beta)} - b(1-\beta)t\Big]^{\frac{1}{1-\beta}}$$

as $t \downarrow 0$, $x = \eta_l(t) = -lt^{\frac{1}{\alpha(1-\beta)}}$

Proof of Lemma 4

$$\forall \epsilon > 0,\ \exists x_\epsilon < 0 \quad \Big(C - \frac{\epsilon}{2}\Big)(-x)_+^\alpha \le u_0(x) \le \Big(C + \frac{\epsilon}{2}\Big)(-x)_+^\alpha,\ x \ge x_\epsilon$$

$$\exists \delta > 0 \ \ u_{-\epsilon}(x_\epsilon, t) \le u(x_\epsilon, t) \le u_\epsilon(x_\epsilon, t), \quad 0 \le t \le \delta$$

Let $u_{\pm\epsilon}$ be a solution of the problem

$$\begin{cases} v_t - (v^m)_{xx} + bv^\beta = 0,\ |x| < |x_\epsilon|, 0 < t \le \delta \\ v(x,0) = (C \pm \epsilon)(-x)_+^\alpha,\ |x| \le |x_\epsilon| \\ v(x_\epsilon, t) = (C \pm \epsilon)(-x_\epsilon)^\alpha,\ v(-x_\epsilon, t) = u(-x_\epsilon, t),\ 0 \le t \le \delta \end{cases}$$

Comparison Theorem implies

$$u_{-\epsilon} \le u \le u_\epsilon, \quad |x| \le |x_\epsilon|,\ 0 \le t \le \delta$$

Proof of Lemma 4

Scale $u_{\pm\epsilon}$ according to inv. scale of $u_t + bu^\beta = 0$, $u(x,0) = C(-x)_+^\alpha$:

$$u_k(x,t) = ku\big(k^{\gamma_1}x, k^{\gamma_2}t\big),$$

$$u_k(x,0) = kC(-k^{\gamma_1}x)^\alpha = k^{1+\gamma_1\alpha}C(-x)_+^\alpha = k^{1+\gamma_1\alpha}u(x,0)$$

$$u_{k,t} + bu_k^\beta = k^{1+\gamma_2}u_t + bk^\beta u^\beta$$

Hence if we choose

$$1 + \gamma_1\alpha = 0,\ 1 + \gamma_2 = \beta \quad \text{or } \gamma_1 = -1/\alpha, \gamma_2 = \beta - 1$$

Then by uniqueness

$$u_k(x,t) = u(x,t) = ku\big(k^{-1/\alpha}x, k^{\beta-1}t\big).$$

Proof of Lemma 4

Set

$$u_k^{\pm\epsilon}(x,t) = k u_{\pm\epsilon}\big(k^{-1/\alpha}x, k^{\beta-1}t\big)$$

$$(u_k^{\pm\epsilon})_t + b(u_k^{\pm\epsilon})^\beta = k^\beta (u_{\pm\epsilon})_\tau + bk^\beta u_{\pm\epsilon}^\beta = k^\beta \left[(u_{\pm\epsilon})_\tau + bu_{\pm\epsilon}^\beta\right] =$$
$$= k^\beta \left(u_{\pm\epsilon}^m\right)_{XX} = k^\beta k^{-m+\frac{2}{\alpha}} (u_k^{\pm\epsilon})_{xx}^m$$

So $u_k^{\pm\epsilon}(x,t)$ satisfies

$$\begin{cases} v_t - k^{\frac{2-\alpha(m-\beta)}{\alpha}}(v^m)_{xx} + bv^\beta = 0 \text{ in } E_\epsilon^k \\ v(x,0) = (C \pm \epsilon)(-x)_+^\alpha, \ |x| \le k^{\frac{1}{\alpha}}|x_\epsilon| \\ v\big(k^{\frac{1}{\alpha}}x_\epsilon, t\big) = k(C \pm \epsilon)(-x_\epsilon)^\alpha \\ v(-k^{\frac{1}{\alpha}}x_\epsilon, t) = ku(-x_\epsilon, k^{\beta-1}t), \ 0 \le t \le k^{1-\beta}\delta \end{cases}$$

$$E_\epsilon^k = \{|x| < k^{\frac{1}{\alpha}}|x_\epsilon|, \quad 0 < t \le k^{1-\beta}\delta\}$$

Note: $\frac{2-\alpha(m-\beta)}{\alpha} < 0$.

Proof of Lemma 4

Convergence of $\{u_k^{\pm\epsilon}\}$: Take $g(x,t) = (C+1)(1+x^2)^{\alpha/2}e^t$.

$$\tilde{L}_k g = g_t - k^{\frac{2-\alpha(m-\beta)}{2}}(g^m)_{xx} + bg^\beta \geq g(1+R) \text{ in } E_{o\epsilon}^k = E_\epsilon^k \cap \{0 < t \leq t_0\}$$

$$R = O(k^\theta) \text{ uniformly for } (x,t) \in E_{o\epsilon}^k \text{ as } k \to \infty,$$

$$\theta = \begin{cases} \frac{2-\alpha(m-\beta)}{\alpha}, & \text{if } \alpha < \frac{2}{m-1} \\ \beta - 1, & \text{if } \alpha \geq \frac{2}{m-1} \end{cases}$$

Proof of Lemma 4

$$0 < \epsilon \ll 1 \Rightarrow g(x,0) \geq u_k^{\pm\epsilon}(x,0), \ |x| \leq k^{\frac{1}{\alpha}}|x_\epsilon|$$

Since

$$u_k^{\pm\epsilon}(-k^{\frac{1}{\alpha}}x_\epsilon, t) = o(k), \ 0 \leq t \leq t_0 \text{ as } k \to \infty,$$

$$g(\pm k^{\frac{1}{\alpha}}x_\epsilon, t) \geq u_k^{\pm\epsilon}(\pm k^{\frac{1}{\alpha}}x_\epsilon, t), \ 0 \leq t \leq t_0$$

If $k \gg 1$ by Comparison Theorem we have

$$0 \leq u_k^{\pm\epsilon}(x,t) \leq g(x,t) \text{ in } \bar{E}_{o\epsilon}^k$$

Hence, $\{u_k^{\pm\epsilon}\}$ is uniformly bounded in $\bar{E}_{o\epsilon}^k$ and accordingly ($[UA, JDE]$) it is uniformly Hölder continuous on any compact $G \subset P = \{x \in \mathbb{R}, 0 < t \leq t_0\}$ s.t.$\Rightarrow \exists k'$ such that

$$\lim_{k' \to +\infty} u_{k'}^{\pm\epsilon}(x,t) = v_{\pm\epsilon}(x,t) \text{ which solves}$$

Proof of Lemma 4

$$v_t + bv^\beta = 0,\ x \in \mathbb{R},\ 0 < t \le t_0;\ v(x,0) = (C \pm \epsilon)(-x)_+^\alpha,\ x \in \mathbb{R}$$

$$v_{\pm\epsilon}(x,t) = \Big[(C \pm \epsilon)^{1-\beta}(-x)_+^{\alpha(1-\beta)} - b(1-\beta)t\Big]_+^{\frac{1}{1-\beta}}$$

Let

$$l > l_* = C^{-\frac{1}{\alpha}}\big(b(1-\beta)\big)^{\frac{1}{\alpha(1-\beta)}}, \quad \text{and } \epsilon > 0 \text{ are chosen such that}$$

$$(C-\epsilon)^{1-\beta} l^{\alpha(1-\beta)} > b(1-\beta)$$

Take $x = \eta_l(t) = -lt^{\frac{1}{\alpha(1-\beta)}}$.

$$\lim_{k \to +\infty} k u_{\pm\epsilon}\big(-lk^{-\frac{1}{\alpha}} t^{\frac{1}{\alpha(1-\beta)}}, k^{\beta-1}t\big) = v_{\pm\epsilon}(-lt^{\frac{1}{\alpha(1-\beta)}}, t)$$

Proof of Lemma 4

Let $\tau = k^{\beta-1}t$ so that $\tau \to 0$ as $k \to +\infty$; then

$$k^{-\frac{1}{\alpha}} t^{\frac{1}{\alpha(1-\beta)}} = \left(k^{\beta-1}t\right)^{\frac{1}{\alpha(1-\beta)}} = \tau^{\frac{1}{\alpha(1-\beta)}}$$

$$v_{\pm\epsilon}(-lt^{\frac{1}{\alpha(1-\beta)}}, t) = \left[(C \pm \epsilon)^{1-\beta} l^{\alpha(1-\beta)} - b(1-\beta)\right]^{\frac{1}{1-\beta}} t^{\frac{1}{1-\beta}} \Rightarrow$$

$$\lim_{\tau \downarrow 0} \tau^{\frac{1}{\beta-1}} u_{\pm\epsilon}(-l\tau^{\frac{1}{\alpha(1-\beta)}}, \tau) = \left[(C \pm \epsilon)^{1-\beta} l^{\alpha(1-\beta)} - b(1-\beta)\right]^{\frac{1}{1-\beta}}$$

$$u_{\pm\epsilon}(\eta_l(\tau), \tau) \sim \left[(C \pm \epsilon)^{1-\beta} l^{\alpha(1-\beta)} - b(1-\beta)\right]^{\frac{1}{1-\beta}} \tau^{\frac{1}{1-\beta}}, \ \tau \to 0^+$$

By comparison theorem, $u_{-\epsilon} \leq u \leq u_{\epsilon}$ for $x \geq x_{\epsilon}$, $0 \leq t \leq \delta$ hence

$$u(x,t) \sim \left[C^{1-\beta}(-x)_+^{\alpha(1-\beta)} - b(1-\beta)t\right]^{\frac{1}{1-\beta}}$$

as $t \downarrow 0^+$ along $x = \eta_l(t)$. Lemma is proved.