LECTURE 4
INTRODUCTION TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS IV. NONLINEAR DIFFUSION IN NON-SMOOTH DOMAINS

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In this lecture I am going to talk about nonlinear diffusion equation in general non-smooth domains. In a series of papers I developed the theory of nonlinear diffusion type equations in general non-smooth domains. In this talk I am going to explain the results of my papers [1, 2]. A particular motivation for this works arises from the problem about the evolution of interfaces in problems for porous medium equation. Special interest concerns the cases when support of the initial data contains a corner or cusp singularity at some points. What about the movement of these kind of singularities along the interface? To solve this problem, it is important at the first stage to develop general theory of boundary-value problems in non-cylindrical domains with boundary surfaces which has the same kind of behaviour as the interface. In many cases this may be non-smooth and characteristic. It should be mentioned that in the one-dimensional case Dirichlet and Cauchy-Dirichlet problems for the reaction-diffusion equations in irregular domains were studied in [3, 4]. Primarily applying this theory a complete description of the evolution of interfaces were presented in [5, 6].

1. Dirichlet Problem in General Non-smooth Domains

Let me introduce some notation before formulating the Dirichlet problem for the nonlinear diffusion equation. Let Ω be a bounded open subset of $\mathbb{R}^{N+1}$, $N \geq 2$. Let the boundary $\partial\Omega$ consist of the closure of a domain $B\Omega$ lying on $\{t = 0\}$, a domain $D\Omega$ lying on $\{t = T\}, T \in (0, +\infty)$ and a (not necessarily connected) manifold $S\Omega$ lying in the strip $0 < t \leq T$. denote

$$\Omega(\tau) = \{(x, t) \in \Omega : t = \tau\}$$

and assume that $\Omega(t) \neq \emptyset$ for $t \in (0, T)$. I shall use standard notation:

$$z = (x, t) = (x_1, ..., x_N, t) \in \mathbb{R}^{N+1}, N \geq 2, x = (x_1, \overline{x}) \in \mathbb{R}^N, \overline{x} = (x_2, ..., x_N) \in \mathbb{R}^{N-1}$$

For a point $z = (x, t) \in \mathbb{R}^{N+1}$, I denote by $B(z; \delta)$ an open ball in $\mathbb{R}^{N+1}$ of radius δ and with center in z.

Assume that for an arbitrary point $z_0 = (x^0, t_0) \in S\Omega$ (or $z_0 = (x^0, 0) \in \overline{S\Omega}$) there exists $\delta > 0$ and a continuous function ϕ such that, after a suitable rotation of x-axes, we have

$$\overline{S\Omega} \cap B(z_0, \delta) = \{z \in B(z_0, \delta) : x_1 = \phi(\overline{x}, t)\}$$

$$\text{sign}(x_1 - \phi(\overline{x}, t) = 1, \text{for } x \in B(z_0, \delta) \cap \Omega.$$

The set $\mathcal{P}\Omega = \overline{B\Omega} \cup S\Omega$ is called a parabolic boundary of Ω. Furthermore the class of domains with described structure will be denote by $\mathcal{D}_{0,T}$.

Let $\Omega \in \mathcal{D}_{0,T}$ be given and let ψ be an arbitrary continuous non-negative function defined on $\mathcal{P}\Omega$. Here is the formulation of the *Dirichlet Problem* (DP): find the solution $u(x,t)$ from

$$u_t = \Delta\, u^m, \quad \text{in } \Omega \cup D\Omega \tag{1}$$

$$u = \psi, \quad \text{on } \mathcal{P}\Omega \tag{2}$$

where $m > 0$. I am going to follow the following notion of weak solution (super- or subsolution):

Definition 1. We say that a function $u = u(x,t)$ is a solution (respectively, super- or subsolution) of DP (1), (2) if

(1) $u \geq 0$
(2) u is continuous in $\overline{\Omega}$, locally Hölder continuous in $\Omega \cup D\Omega$, satisfying (2) (respectively satisfying (2) with $=$ replaced by $\geq$ or $\leq$).
(3) for any t_0, t_1 such that $0 < t_0 < t_1 \leq T$ and for any domain $\Omega_1 \in \mathcal{D}_{t_0,t_1}$ such that $\overline{\Omega}_1 \subset \Omega \cup D\Omega$ and $\partial B\Omega_1$, $\partial D\Omega_1$, $S\Omega_1$ being sufficiently smooth manifolds, the following integral identity holds:

$$\int_{D\Omega_1} uf\,dx = \int_{B\Omega_1} uf\,dx + \int_{\Omega_1} (uf_t + u^m\,\Delta\, f)\,dx\,dt - \int_{S\Omega_1} u^m \frac{\partial f}{\partial \nu}\,dx\,dt \tag{3}$$

(respectively (3) holds with $=$ replaced by $\geq$ or $\leq$), where $f \in C^{2,1}_{x,t}(\overline{\Omega}_1)$ is an arbitrary function (respectively non-negative function) that equals zero on $S\Omega_1$ and ν is the outward-directed normal vector to $\Omega_1(t)$ at $(x,t) \in S\Omega_1$.

2. Boundary Regularity and Existence

As you can see, for a while there is no condition on the regularity of the boundary manifold. It is just locally continuous manifold. My goal is to prove well-posedness of the DP under minimal restrictions on the boundary manifold. I am going now to formulate a pointwise condition on the lateral boundary, introduced in [2], for the boundary regularity of the solution of the DP.

Let $z_0 = (x^0, t_0) \in S\Omega$ be a given boundary point. For an arbitrary sufficiently small $\delta > 0$, consider a domain

$$P(\delta) = \{(\overline{x}, t) : |\overline{x} - \overline{x}^0| < (\delta + t - t_0)^{\frac{1}{2}}, t_0 - \delta < t < t_0\}$$

Definition 2. The function

$$\omega(\delta) = \max\left(\phi(\overline{x}^0, t_0) - \phi(\overline{x}, t) : (\overline{x}, t) \in \overline{P(\delta)}\right)$$

is called *the parabolic modulus of left-lower semicontinuity* of the function ϕ at the point $(\overline{x}^0, t_0)$.

For sufficiently small $\delta > 0$ the function $\omega(\delta)$ is well defined and converge to zero as $\delta \downarrow 0$. Let me know formulate assumption $\mathcal{A}$ introduced in [2].

Assumption $\mathcal{A}$. There exists a function $F(\delta$ which is defined for all positive sufficiently small δ; F is positive with $F(\delta) \to 0+$ as $\delta \downarrow 0$ and

$$\omega(\delta) \leq \delta^{\frac{1}{2}} F(\delta). \tag{4}$$

It is proved in [2] that Assumption $\mathcal{A}$ is sufficient for the regularity of the boundary point z_0. Namely, the solution u takes boundary value $\psi(z_0)$ at the point $z = z_0$ continuously in $\overline{\Omega}$.

Denote $x_1 = \overline{\phi}(\overline{x}) \equiv \phi(\overline{x}, 0)$.

Definition 3. The function

$$\omega_0(\delta) = \max\Big(\phi(\overline{x}^0) - \phi(\overline{x}) : |\overline{x} - \overline{x}^0| \leq \delta\Big)$$

is called *the modulus of lower semicontinuity* of the function $x_1 = \overline{\phi}(\overline{x})$ at the point $\overline{x} = \overline{x}^0$.

Assumption $\mathcal{B}$. There exists a function $F_1(\delta$ which is defined for all positive sufficiently small δ; F_1 is positive with $F_1(\delta) \to 0+$ as $\delta \downarrow 0$ and

$$\omega_0(\delta) \leq \delta F_1(\delta). \tag{5}$$

It is proved in [2] that Assumption $\mathcal{B}$ is sufficient for the regularity of the boundary point $z_0 = (x^0, 0) \in \overline{S\Omega}$.

The following existence and boundary regularity result is proved in [2]:

Theorem 1. [2]. *DP is solvable in a domain Ω which satisfies Assumption $\mathcal{A}$ at every point $z_0 \in S\Omega$ and Assumption $\mathcal{B}$ at every point $z_0 = (x^0, 0) \in \overline{S\Omega}$.*

In fact, solution is smooth whenever it is positive.

Corollary 2. *Let the conditions of Theorem 1 be satisfied and $\inf_{\mathcal{P}\Omega} \psi > 0$. then there exists a unique classical solution $u \in C(\overline{\Omega}) \cap C^\infty(\Omega \cup D\Omega)$ of the DP.*

3. Uniqueness, Comparison and Stability Results

Furthermore, I am going to assume that the condition of Theorem 1 is satisfied. Now I am going to formulate another pointwise restriction at the point $z_0 = (x^0, t_0) \in S\Omega, 0 < t_0 < T$, which plays a crucial role in the proof of uniqueness of the solution. For an arbitrary sufficiently small $\delta > 0$, consider a domain

$$Q(\delta) = \{(\overline{x}, t) : |\overline{x} - \overline{x}^0| < (\delta + t_0 - t)^{\frac{1}{2}}, t_0 < t < t_0 + \delta\}$$

Assumption $\mathcal{M}$. Assume that for all positive sufficiently small δ we have

$$\phi(\overline{x}^0, t_0) - \phi(\overline{x}, t) \leq \Big(t - t_0 + |\overline{x} - \overline{x}^0|^2\Big)^\mu, \text{for } (\overline{x}, t) \in \overline{Q(\delta)} \tag{6}$$

where $\mu > \frac{1}{2}$ if $0 < m < 1$, and $\mu > \frac{m}{m+1}$ if $m > 1$.

Assumption $\mathcal{M}$ is of geometric nature. It is pointwise and the related number μ in (6) depends on $z_0 \in S\Omega$ and may vary for different points $z_0 \in S\Omega$.

Definition 4. Let $[c, d] \in (0, T)$ be a given segment and

$$S\Omega_{[c,d]} = S\Omega \cap \{(x, t) : c \leq t \leq d\}$$

We shall say that assumption $\mathcal{M}$ is satisfied uniformly in $[c, d]$ if there exists $\delta_0 > 0$ and $\mu > 0$ as in (6) such that for $0 < \delta < \delta_0$, (6) is satisfied for all $z_0 \in S\Omega_{[c,d]}$ with the same μ.

In [1] I proved the following uniqueness result:

Theorem 3. [1] *Assume that there exists a finite number of points $t_i, i = 1, ..., k$, such that $t_1 = 0 < t_2 < ... < t_k < t_{k+1} = T$ and for the arbitrary compact subsegment $[\delta_1, \delta_2] \subset (t_i, t_{i+1})$, $i = 1, ..., k$, Assumption $\mathcal{M}$ is uniformly satisfied in $[\delta_1, \delta_2]$. Then the solution of the DP is unique.*

Note that along its intersection with finitely many hyperplanes $t = \text{const.}$ (including $t = 0$), boundary manifold is free from any assumption.

Under the same conditions comparison theorem is valid ([2]): let u be a solution of DP, and let g be a supersolution (respectively subsolution) of DP. Then

$$u \leq \text{ (respectively } \geq) \; g \text{ in } \overline{\Omega}$$

In particular, the following stability or L_1-contraction estimation is true:

Theorem 4. [1] *Let the assumptions of Theorem 3 be satisfied. Let g_1 and g_2 be solutions of DP with initial boundary data ψ_1 and ψ_2, respectively. If $\psi_1 = \psi_2 = \psi$ on $S\Omega$, then for arbitrary $t \in [0, T]$ we have*

$$\|g_1 - g_2\|_{L_1(\Omega(t))} \leq \|\psi_1 - \psi_2\|_{L_1(B\Omega)}$$

Corollary 5. *Let the assumptions of Theorem 4 are satisfied. Then for every t and τ such that $0 \leq \tau \leq t \leq T$ we have*

$$\|g_1 - g_2\|_{L_1(\Omega(t))} \leq \|g_1 - g_2\|_{L_1(\Omega(\tau))}$$

Theorem 4 presents a continuous dependence of the solution to DP on the initial data. concerning continuous dependence on the initial-boundary data the following result is true:

Corollary 6. [1] *Let the assumptions of Theorem 3 be satisfied. Let u be a solution of DP. Assume that $\{\psi_n\}$ is a sequence of non-negative continuous functions defined $\mathcal{P}\Omega$ and*

$$\lim_{n\to\infty} \psi_n(z) = \psi(z)$$

uniformly for $z \in \mathcal{P}\Omega$. Let u_n be a solution of DP with $\psi = \psi_n$. Then

$$\lim_{n\to\infty} u_n = u \text{ in } \overline{\Omega}$$

and convergence is uniform on compact subsets of $\Omega \cup D\Omega$.

4. Boundary Regularity for the Linear Diffusion/Heat Equation

Note that linear diffusion equation ($m = 1$) is a particular case of the PDE (1). Assumption $\mathcal{A}$ for the boundary regularity can be significantly sharpened for the linear case. In particular, boundary point is regular for the heat equation if Assumption $\mathcal{A}$ is satisfied with $F(\delta)$ being replaced with arbitrary positive constant $M > 0$. In fact, consider a hyperbolic paraboloid

$$x_1^2 = M(-t + |\overline{x}|^2)$$

and for given small $\delta > 0$, cut the subsurface of the hyperbolic paraboloid which is situated in the half-space $\{t \leq 0\}$ between two planes $\{x_1 = 0\}$ and $\{x_1 = -\delta^{\frac{1}{2}}\}$. Let M_δ be open set bounded by this subsurface and by the planes $\{t = 0\}$ and $\{x_1 = -\delta^{\frac{1}{2}}\}$. This subsurface of the hyperbolic paraboloid satisfies the Assumption $\mathcal{A}$ with constant $M > 0$ instead of $F(\delta)$. In my paper [7] it is proven that the origin is a regular boundary point for the heat equation in a region which is a complement of M_δ. This result implies geometric boundary regularity test for the heat equation: consider the rigid body displacements of M_δ composed of translations and (or) rotations in x-space and shift along the t-axis. If after such a displacement the vertex M_δ coincides with the boundary point, and for all sufficiently small $\delta > 0$ it lies outside of the domain, then boundary point is regular.

In fact, optimal result in terms of dependence of the boundary regularity for the heat equation on the geometry and regularity of the exterior touching surface requires the function

$F(\delta)$ in Assumption $\mathcal{A}$ to grow to infinity as $\delta \downarrow 0$. Let me now describe almost optimal result, so called geometric *iterated logarithm test* from my paper [7].

Consider the following domains

$$\mathcal{G}^1_\rho = \{z : x_1^2 < 4\xi \log \rho(\xi), -\delta < \xi < 0, -\delta < \alpha t < 0\}$$

$$\mathcal{G}^2_\rho = \{z : -2(\xi \log \rho(\xi))^{\frac{1}{2}} < x_1 < 2(-\delta \log \rho(-\delta))^{\frac{1}{2}}, -\delta < \xi < 0, -\delta < \alpha t < 0\}$$

where $\delta > 0$ is a sufficiently small positive number, $\xi = \alpha t - \beta|\overline{x}|^2$, and α, β are positive numbers; $\rho = \rho(\xi) \in C[-\delta, 0)$ satisfies $\rho(\xi) \downarrow 0$ as $\xi \uparrow 0$.

The main result of [7] says:

Theorem 7. [7] *Let* $\alpha + 2(N-1)\beta \leq 1$ *and*

$$\int^{0-} \frac{\rho(\eta)}{\eta} d\eta = -\infty$$

Then the origin is a regular point for $\mathcal{G}^1_\rho$ *and* $\mathcal{G}^2_\rho$

Example of the function ρ which satisfy the conditions of the Theorem 7 is the following iterated logarithm function:

$$\rho(\xi) = \left\{ |\log|\xi|| \prod_{k=2}^{n} \log_k |\xi| \right\}^{-1} \tag{7}$$

where $n \geq 2$ be an arbitrary integer, and lower index k means the number of iterations of the logarithm.

Theorem 7 implies general geometric iterated logarithm test. Let $\mathcal{A}_\rho = \mathcal{G} \setminus \overline{\mathcal{G}^2_\rho}$, where

$$\mathcal{G} = \{z : x_1^2 < -4\delta \log \rho(-\delta), -\delta < \xi < 0, -\delta < \alpha t < 0\}$$

We call the origin the vertex of $\mathcal{A}_\rho$. Consider the rigid body displacements of $\mathcal{A}_\rho$ composed of translations and (or) rotations in x-space and shift along the t-axis.

Definition 5. We shall say that Ω satisfies the exterior $\mathcal{A}_\rho$-condition at the point $z_0 \in S\Omega$ if after the above mentioned displacement the vertex of $\mathcal{A}$ coincides with z_0 and for all sufficiently small δ, $\mathcal{A}_\rho$ lie in the exterior of Ω.

In [7] it is proved that the boundary point $z_0 \in S\Omega$ is regular if Ω satisfies the exterior $\mathcal{A}_\rho$ condition at the boundary point z_0. This result is quite sharp. In particular, the origin is and irregular boundary point for $\mathcal{G}^1_\rho$ and $\mathcal{G}^2_\rho$ if ρ is chosen according to (7) but with $\alpha > 1$. Another important consequence is that the origin may be regular for

$$u_t = \Delta u, \tag{8}$$

and at the same time irregular for

$$u_t = a \Delta u, \ 0 < a < 1 \tag{9}$$

regarded as a boundary point of $\mathcal{G}^1_\rho$ and $\mathcal{G}^2_\rho$. Let us check this fact by considering

$$\rho(\xi) = |\log|\xi||^{-1}, \ \alpha = 1 - \frac{\epsilon}{2}, \ \beta = \frac{\epsilon}{4(N-1)}$$

where ϵ is an arbitrary number satisfying $0 < \epsilon \leq 1 - a$. From Theorem 7 it follows that the origin is regular for the heat equation (8). If $u(x,t)$ solves the equation (9), then after transformation $\tau = at$, the function $\tilde{u}(x,\tau) = u(x,t)$ satisfies

$$\tilde{u}_\tau = \Delta \tilde{u}, \tag{10}$$

while the domain $\mathcal{G}^1_\rho$ is transformed to the domain

$$\tilde{\mathcal{G}}^1_\rho = \{z = (x,\tau) : x_1^2 < 4\xi_1 \log \rho(\xi_1), -\delta < \xi_1 < 0, -\delta < \alpha a^{-1}\tau < 0\},$$

where $\xi_1 = \alpha a^{-1}\tau - \beta|\overline{x}|^2$. Since $\alpha a^{-1} > (1-\epsilon)a^{-1} \geq 1$, the origin is irregular for (10). Accordingly, the origin is irregular for (9) regarded as a boundary point of $\mathcal{G}^1_\rho$ and $\mathcal{G}^2_\rho$.

Finally, let me describe the necessary and sufficient condition from my paper [8] for the regularity of the characteristic boundary point for domains of special structure:

$$\Omega_\delta = \{(x,t) \in \mathbb{R}^{N+1} : |x| < h(t), -\delta < t < 0\}$$

where $h \in C[-\delta, 0]$, $h > 0$ for $t < 0$ and $h(t) \downarrow 0$ as $t \uparrow 0$. Let us choose

$$h(t) = 2(t \log \rho(t))^{\frac{1}{2}}$$

and assume that $\rho \in C[-\delta, 0]$, $\rho(t) > 0$ for $-\delta \leq t < 0$; $\rho(t) \downarrow 0$ as $t \uparrow 0$.

Theorem 8. [8] *The origin is regular or irregular as a boundary point of Ω_δ regarding heat equation according as*

$$\int^{0-} \frac{\rho(t)|\log \rho(t)|^{\frac{N}{2}}}{t} dt$$

diverges or converges.

For example, if the boundary near the origin is given by

$$|x|^2 = -4t\Big(\log|\log|t|| + \frac{N+2}{2}\log_3|t| + \cdots + (1+\epsilon)\log_n|t|\Big)$$

where $n \geq 4$ be any integer, then origin is regular or irregular according to whether $\epsilon = 0$ or $\epsilon > 0$. If space dimension $N = 1$, then the result of Theorem 8 coincides with Petrovsky's celebrated result from [9].

In fact, a particular motivation for the consideration of the domain Ω_δ is the problem about the local asymptotic behaviour of the Brownian motion trajectories for the diffusion processes. Let me describe the probabilistic counterpart of theorem 8 in the context of the multidimensional Brownian motion. Consider the standard N-dimensional Brownian motion

$$\mathcal{D} = [\xi(t) = (x_1(t), x_2(t), \cdots, x_N(t) : t \geq 0, P_\bullet]$$

in which the coordinates of the sample path are independent standard 1-dimensional Brownian motions and $P_\bullet(B)$ is the probability of B as a function of the starting point $\xi(0)$ of the N-dimensional Brownian path. Consider the radial part

$$r(t) = (x_1^2(t) + \cdot + x_N^2(t))^{\frac{1}{2}}$$

of the standard N-dimensional Brownian path. Blumenthal's 0-1 law implies that $P_0[r(t) < h(t), t \downarrow 0] = 0$ or 1; h is said to belong to the upper class if this probability is 1 and to the lower class otherwise. The probabilistic counterpart of Theorem 8 states that if $h \in \uparrow$ and if $t^{-\frac{1}{2}}h \in \downarrow$ for small $t > 0$, then h belongs to the upper class or to the lower class according as

$$\int_{0+} t^{-\frac{N}{2}-1} h^N(t) \exp\Big(-\frac{h^2}{2t}\Big) dt$$

converges or diverges. When $N = 1$, this is the well-known Kolmogorov-Petrovsky test.

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