LECTURE 3
INTRODUCTION TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS III. ENERGY ESTIMATES AND EXISTENCE

UGUR G. ABDULLA

In this talk, I am going to prove existence of the weak solution of the Dirichlet problem for the nonlinear diffusion equation under minimal restrictions on the data. Let me remind you what I did in previous lectures. I considered the following Dirichlet problem:

$$u_t = \Delta u^m, \quad \text{in } Q_T = \Omega \times (0,T] \tag{1}$$

$$u(x,0) = u_0(x), \quad \text{in } \Omega \tag{2}$$

$$u(x,t) = 0, \quad \text{on } S_T = \partial\Omega \times (0,T] \tag{3}$$

In the first lecture I constructed instantaneous point source solution, or so called ZKB-solution of the nonlinear diffusion equation. This solution exhibits finite speed of propagation property which is in sharp contrast to classical linear heat/diffusion equation. In fact, ZKB solution is not a solution in a classical sense, and even first order derivatives are discontinuous along the interfaces or free boundaries separating the regions where solution is positive from the region where it is zero. To justify this important and physically relevant solution, we introduced the notion of the weak solution. Recall the definition of the weak solution to the Dirichlet problem:

Definition 1. We say that a function $u = u(x,t)$ is a weak solution of the Dirichlet problem for the Nonlinear Diffusion Equation, (1)–(3) if

(1) $u \geq 0$
(2) $u^m \in L_2(0,T; H_0^1(\Omega))$,
(3) u satisfies the integral identity

$$\iint_{Q_T} (\nabla u^m \cdot \nabla \phi - u\phi_t) \, dx\, dt = \int_\Omega u_0(x)\phi(x,0)\, dx$$

for any $\phi \in C^1(\overline{Q}_T)$ satisfying $\phi(x,T) = \phi\Big|_{S_T} = 0$.

In particular, ZKB solution is a weak solution. In the previous lecture, I proved a uniqueness of the weak solution of the Dirichlet problem with initial data in $L_1(\Omega)$. In particular, this implies that if initial function is chosen according to ZKB solution, then classical solution doesn't exist and ZKB solution is the only solution.

My next goal is to prove that the weak solution exists for a general class of initial functions, not necessarily specific ones for ZKB solution. The proof of the existence will be based on the energy estimates. The main difficulty to prove the existence result is that the PDE (1) is a nonlinear *degenerate* parabolic equation. Let me write PDE (1) in a divergence form:

$$u_t = \operatorname{div}(mu^{m-1}\nabla u) \tag{4}$$

Diffusion or heat conduction coefficient is mu^{m-1}, and it vanishes when $u = 0$. Hence, at all those points where $u = 0$ equation looses parabolicity, or degenerates, and accordingly classical theory of uniformly parabolic equations [1] is not applicable. Moreover, it is an implicit degeneration, since it depends on the unknown solution, and we don't know at what part of the domain it will happen.

The main idea of the proof is based on regularization: approximate initial and boundary functions with the sequence of smooth and positive functions; use classical parabolic theory and construct the sequence of smooth classical solutions of the problem with regularized data; derive uniform energy estimate for the sequence of classical solutions; Use compactness result and derive weak solution of the original problem as a limit of the sequence of classical solutions corresponding to regularized data. The first result of this nature appeared in [2], and for modern treatment I refer to [3].

I am going to pursue the proof of the existence in several steps. Required minimal condition on the initial function u_0 will be deduced along the way.

(1) First assume $u_0 \in C_c^\infty(\Omega)$, that is to say it is infinitely differentiable in Ω and vanishes near the boundary $\partial\Omega$. The regularizing step consists in taking

$$u_{0,n}(x) = u_0(x) + \frac{1}{n}$$

and considering the problem

$$\begin{cases} u_t = \operatorname{div}\left(mu^{m-1}\nabla u\right) & \text{in } Q_T \\ u(x,0) = u_{0,n}(x) & \text{in } \Omega \\ u(x,t) = \frac{1}{n} & \text{on } S_T \end{cases} \tag{5}$$

Now, the boundary and initial values of the problem are strictly positive, and hence the diffusion coefficient can be expected to be also strictly positive in Q_T. The classical theory of uniformly parabolic PDEs (see for example [1]) implies that there exists a unique solution $u_n(x,t)$ to (5), such that $u_n(x,t) \in C^{2,1}(\bar{Q}_T) \cap C^\infty(Q_T)$ and it satisfies a maximum principle: the maximum and minimum values of the solution are taken on the parabolic boundary $\Omega \cup S_T$. Hence, we have

$$\frac{1}{n} \le u_n(x,t) \le M + \frac{1}{n}$$

where $M = \max_\Omega u_0$. Maximum principle also implies that the classical solution is order preserving, meaning if one solution dominates over the other solution on the parabolic boundary $\Omega \cup S_T$, then domination is preserved everywhere in Q_T. Accordingly, we have

$$u_n(x,t) \ge u_{n+1}(x,t) \ge \cdots \ge 0$$

Since this is a bounded monotonically decreasing sequence of functions, it has a pointwise limit; set

$$u(x,t) = \lim_{n\to\infty} u_n(x,t), \quad (x,t) \in \bar{Q}_T$$

We hope that u is a solution of (1)–(3), with the given initial data. But this is where the difficulty arises: a pointwise limit of C^∞ functions may not even be continuous! In particular, its limit may not have the appropriate weak derivatives required by definition 1.

(2) We will show now how the differential operator (1) will in fact furnish the required regularity for the limit function. The first question is: can we control ∇u_n^m?

Let's write down the equation satisfied by u_n, multiplied by a test function Φ, and integrated over Q_T:

$$0 = \iint_{Q_T} \Big((u_n)_t - \Delta\, u_n^m\Big)\Phi \, dx\, dt$$

Choose $\Phi = u_n^m - \frac{1}{n^m}$:

$$0 = \iint_{Q_T} \Big((u_n)_t - \Delta\, u_n^m\Big)\left(u_n^m - \frac{1}{n^m}\right) dx\, dt$$

The idea here is exactly that when we integrate by parts with respect to x, we will have $|\nabla u^m|^2$, while still having $\Phi = 0$ on S_T.

Integrating by parts (i.e. using Gauss-Green,) we have

$$0 = \iint_{Q_T} |\nabla u_n^m|^2 \; dx\, dt + \iint_{Q_T} \left[\frac{1}{m+1}\left(u_n^{m+1}\right)_t - \frac{1}{n^m}(u_n)_t\right] dx\, dt$$

It follows that

$$\begin{aligned} 0 = \iint_{Q_T} |\nabla u_n^m|^2 \; dx\, dt + \frac{1}{m+1}\int_\Omega u_n^{m+1}(x,T)\, dx - \\ - \frac{1}{m+1}\int_\Omega u_{0,n}^{m+1}(x)\, dx - \frac{1}{n^m}\int_\Omega u_n(x,T)\, dx + \\ + \int_\Omega \frac{1}{n^m} u_{0,n}(x)\, dx \end{aligned}$$

Let's put the term with the gradient on one side of the equality, together with the positive term with u_n^{m+1}, to derive

$$\iint_{Q_T} |\nabla u_n^m|^2 \; dx\, dt + \frac{1}{m+1}\int_\Omega u_n^{m+1}(x,T)\, dx = \frac{1}{m+1}\int_\Omega u_{0,n}^{m+1}(x)\, dx + \\ + \frac{1}{n^m}\int_\Omega u_n(x,T)\, dx - \int_\Omega \frac{1}{n^m} u_{0,n}(x)\, dx$$

The term with negative sign is ≤ 0, so it can be discarded if we replace $=$ with $\leq$. Look to the other terms present on the RHS: those containing $u_{0,n}$ are good terms, since they are controlled by the data u_0. This will also tell us what are the minimal conditions on u_0 to control the LHS, which in particular includes integral of $|\nabla u^m|^2$. We have

$$\begin{aligned} \iint_{Q_T} |\nabla u_n^m|^2 \; dx\, dt + \frac{1}{m+1}\int_\Omega u_n^{m+1}(x,T)\, dx \leq \\ \leq \frac{1}{m+1}\int_\Omega u_{0,n}^{m+1}(x)\, dx + \frac{1}{n^m}\int_\Omega u_n(x,T)\, dx \end{aligned}$$

Estimate the last integral using $u_n(x,t) \le M + \frac{1}{n}$ to derive

$$\iint_{Q_T} |\nabla u_n^m|^2 \, dx\, dt + \frac{1}{m+1}\int_\Omega u_n^{m+1}(x,T)\, dx \le$$
$$\le \frac{1}{m+1}\int_\Omega u_{0,n}^{m+1}(x)\, dx + \frac{1}{n^m}|\Omega|\left(M + \frac{1}{n}\right)$$

where $|\Omega|$ is a Lebesgue measure of Ω. Now, the first term on the RHS is uniformly bounded, as is the second. So in particular, the L_2 norm of the gradient is indeed controlled by the particular integral norm of the initial function.

(3) Hence $\{\nabla u_n^m\}$ is uniformly bounded in $L_2(Q_T)$:

$$\iint_{Q_T} |\nabla u_n^m|^2 \, dx\, dt \le C \text{ uniformly with respect to } n$$

At this point, we refer to a functional analysis result [4]: uniformly bounded sequence in Hilbert space $L_2(Q_T)$ is weakly compact, meaning that there exists a sequence n' such that

$$\nabla u_{n'}^m \xrightarrow{w} \boldsymbol{\psi} \in L_2^n(Q_T) \text{ as } n' \to \infty$$

where $\xrightarrow{w}$ means weak convergence; $\boldsymbol{\psi} : Q_T \to \mathbb{R}^n$ be a vector function, $L_2^n(Q_T)$ be a space of measurable vector functions $\mathbf{v} : Q_T \to \mathbb{R}^n$ such that $|\mathbf{v}| \in L_2(Q_T)$. That is to say, for any $\phi \in L_2(Q_T)$,

$$\lim_{n'\to\infty} \iint_{Q_T} \nabla u_{n'}^m \phi \, dx\, dt = \iint_{Q_T} \boldsymbol{\psi}\phi \, dx\, dt$$

Question: is $\boldsymbol{\psi} = \nabla u^m$?

To answer this question, choose any $\phi \in C_c^\infty(Q_T)$. Integrate by parts on the LHS:

$$\iint_{Q_T} \nabla u_{n'}^m \phi \, dx\, dt = -\iint_{Q_T} u_{n'}^m \nabla\phi \, dx\, dt$$

Pointwise convergence implies in particular L_2 convergence, and hence we can pass to the limit to derive

$$\iint_{Q_T} \boldsymbol{\psi}\phi \, dx\, dt = \lim_{n'\to\infty} \iint_{Q_T} \nabla u_{n'}^m \phi \, dx\, dt =$$
$$= -\lim_{n'\to\infty} \iint_{Q_T} u_{n'}^m \nabla\phi \, dx\, dt = -\iint_{Q_T} u^m \nabla\phi \, dx\, dt$$

By the definition of the weak derivative, indeed, the limit $\boldsymbol{\psi}$ is exactly ∇u^m. But this should be true for any other limit point $\boldsymbol{\psi}'$. Since the weak derivative is unique it follows that the whole sequence converges to ∇u^m weakly in Q_T:

$$\nabla u_n^m \xrightarrow{w} \nabla u^m \text{ as } n \to \infty$$

Now, weak convergence does not imply strong convergence. On the other hand, a result known as the uniform boundedness or resonance theorem from functional analysis [4] implies

$$\|\nabla u^m\|_{L_2(Q_T)} \le \liminf_{n\to+\infty} \|\nabla u_n^m\|_{L_2(Q_T)}$$

(4) We return to the inequality

$$\iint_{Q_T} |\nabla u_n^m|^2 \, dx\, dt + \frac{1}{m+1} \int_\Omega u_n^{m+1}(x,T)\, dx \le$$
$$\le \frac{1}{m+1} \int_\Omega u_{0,n}^{m+1}(x)\, dx + \frac{1}{n^m} \int_\Omega u_n(x,T)\, dx$$

We can take lim inf on both sides of this inequality to derive *the energy estimate*:

$$(m+1) \iint_{Q_T} |\nabla u^m|^2 \, dx\, dt + \int_\Omega u^{m+1}(x,T)\, dx \le \int_\Omega u_0^{m+1}(x)\, dx \tag{6}$$

(5) We will now verify that we have the regularity $u^m \in L_2\Big(0,T; H_0^1(\Omega)\Big)$ as required by definition 1. We only need to show that u has zero trace on the boundary S_T. Recall that

$$u = \lim u_n, \quad 0 \le u \le u_n \text{ in } Q_T$$

on the boundary,

$$0 \le \liminf_{(x,t)\to S_T} u \le \limsup_{(x,t)\to S_T} u \le \frac{1}{n}$$

Taking $n \to \infty$ gives

$$\lim_{(x,t)\to S_T} u(x,t) = 0$$

That is, u has zero trace. This proves that u has the required regularity.

(6) We still have to show that u satisfies the appropriate integral identity. With the preceding results, it is relatively straightforward. Since u_n is a classical solution, it also satisfies the integral identity

$$\iint_{Q_T} (\nabla u_n^m \cdot \nabla \phi - u_n \phi_t)\, dx\, dt = \int_\Omega u_{0,n}(x)\phi(x,0)\, dx$$

for any $\phi \in C^1(\overline{Q}_T)$ satisfying $\phi(x,T) = \phi\Big|_{S_T} = 0$. Passing to the limit as $n \to \infty$, the weak convergence of ∇u_n^m and L_2 convergence of u_n, $u_{0,n}$ gives

$$\iint_{Q_T} (\nabla u^m \cdot \nabla \phi - u \phi_t)\, dx\, dt = \int_\Omega u_0(x)\phi(x,0)\, dx$$

for any $\phi \in C^1(\bar{Q}_T)$ with $\phi\Big|_{S_T} = \phi(x,T) = 0$. The proof of existence is complete for $u_0 \in C_c^\infty(\Omega)$.

(7) Note that the main tool to prove the existence of the weak solution for $u_0 \in C_c^\infty(\Omega)$ was the energy estimate (6). In fact, energy estimate also gives us an indication of the minimal requirement on the initial data under which this argument will work: the energy (the LHS of (6)) is controlled by integral of u_0^{m+1}, which suggests $u_0 \in L_{m+1}(\Omega)$ should be the minimal requirement in the framework of the applied method. I am going to demonstrate this.

First assume that $u_0(x)$ is bounded and vanishes near S_T. That is, assume $u_0 \in L_\infty(\Omega)$; the entire preceding proof goes through in the same way, except you find the corresponding sequence of regularized solutions

$$u_n \in C^{2,1}\Big(Q_T \cup S_T\Big) \cap C^\infty(Q_T)$$

That is, u_n is $C^{2,1}$ up to the lateral boundaries but not up to $\Omega \times \{t = 0\}$. This is natural, since the initial data may not even be continuous. Also the convergence $u_n \to u$ as $t \to 0$ will be in any $L_p(\Omega)$, $1 < p < +\infty$.

(8) We lastly come to the minimal restriction, $u_0 \in L_{m+1}(\Omega)$. Take a sequence of cutoff functions $\{\zeta_k\} \in C_c^\infty(\Omega)$, $0 \le \zeta_k \le 1$, such that $\zeta_k \equiv 1$ at alll points of Ω whose distance from $\partial\Omega$ is greater than, say $1/k$; and $\zeta_k \equiv 0$ in the $1/2k$ neighborhood of $\partial\Omega$. Consider

$$u_{0,k}(x) = \min\{u_0(x)\zeta_k(x), k\}$$

Then each $u_{0,k} \in L_\infty(\Omega)$ and is zero near the boundary. For these functions, we construct weak solution as in previous step. The sequence of solutions $\{u_k(x,t)\}$ satisfies the order preserving principle and hence satisfies

$$0 \le u_k(x,t) \le u_{k+1}(x,t) \le \cdots$$

This is again a monotone sequence, so the pointwise limit $u(x,t)$ exists (it may be infinity somewhere) and to each u_k, the energy estimate (6) applies:

$$(m+1)\iint_{Q_T} |\nabla u_k^m|^2 \, dx\, dt + \int_\Omega u_k^{m+1}(x,T)\, dx \le \int_\Omega u_{0,k}^{m+1}(x)\, dx \tag{7}$$

The RHS here is bounded uniformly with respect to k, since $u_0 \in L_{m+1}(\Omega)$. Hence the LHS is also uniformly bounded. That is,

$$\begin{cases} u_k \in L_\infty\Big(0,\infty; L_{m+1}(\Omega)\Big) \\ \{\nabla u_k^m\} \text{ is unif. bounded in } L_2^n(Q_T) \end{cases}$$

The same compactness argument as before implies that the limit function u posses the same regularity as u_k, and u satisfies the appropriate integral identity, and accordingly u is a solution of the Dirichlet problem. Hence, I proved the following theorem:

Theorem 1. *Let $u_0 \in L_{m+1}(\Omega)$ and $u_0 \ge 0$. Then there exists a solution of the Dirichlet problem.*

Note 1. (1) The general method described in this lecture works for a quite general class of nonlinear degenerate parabolic equations.

(2) The solution constructed in this proof posses the regularity

$$u^m \in L_2\Big(0,t; H_0^1(\Omega)\Big)$$

This doesn't imply pointwise interior continuity of the weak solution. The question of continuity was proved in one dimensional case in [2]. Continuity of the solution of the nonlinear diffusion equation in arbitrary space dimension was first proved in [5]. In fact, solution of the nonlinear diffusion equation is Hölder continuous [6]. Boundary regularity of the solutions to the nonlinear diffusion equation in general non-cylindrical domains with nonsmooth boundaries was proved in [7]. Finally, the theory of nonlinear diffusion equation in general nonsmooth domains was developed in [7].

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E-mail address: abdulla@fit.edu

Department of Mathematical Sciences, Florida Institute of Technology, 150 W. University Blvd., Melbourne, FL 32901