LECTURE 2
INTRODUCTION TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS II. WEAK SOLUTIONS AND UNIQUENESS

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1. Weak Solutions

Let me remind you what I did in a previous lecture. I considered the problem

$$u_t - \operatorname{div}\Big(u^\sigma \nabla u\Big) = 0,\ x \in \mathbb{R}^N,\ t > 0;\ \sigma > 0, \tag{1}$$

$$\int_{\mathbb{R}^N} u(x,t)\,dx = 1,\ t > 0 \tag{2}$$

$$u(x,0) = \delta(x) \tag{3}$$

and constructed instantaneous point-source solution, also called the Zeldovich-Kompaneets-Barenblatt (ZKB) solution.

$$u_*(x,t) = t^{-\frac{N}{2+N\sigma}} \left[\frac{\sigma}{2(2+N\sigma)}\left(\eta_0^2 - \frac{|x|^2}{t^{\frac{2}{2+N\sigma}}}\right)_+\right]^{\frac{1}{\sigma}} \tag{4}$$

Note that the nonlinear diffusion equation is invariant under translation of time and space coordinate, and the solution to the problem

$$\begin{cases} u_t = \operatorname{div}\Big(u^\sigma u\Big), & (x,t) \in \mathbb{R}^N \times \mathbb{R}_+ \\ u(x,0) = u_*(x,T), & x \in \mathbb{R}^N \end{cases}$$

is exactly $u_*(x, t+T)$, which has compact support for all $t \geq 0$.

However, several important questions are left open. First of all, ZKB solution is not a classical solution: it is not even differentiable on the boundary of the support. First important question which we need to answer is the following: In what sense is u_* actually a solution of the problem? Hence, we need to define the notion of the weak solution. Second important question we need to answer is the following: May be there is a smooth solution of the same problem with different properties, and ZKB solution is just physically irrelevant mathematical example. Main goal of this lecture is to answer these important questions.

For clarity, I will concentrate on the one particular problem - Dirichlet problem for the nonlinear diffusion equation. I am going to formulate the notion of the weak solution and prove its uniqueness. Let me note that the pioneering work on the mathematical theory of the nonlinear diffusion type equations is [1]. For complete review and modern treatment I refer to [2].

Consider the Dirichlet problem for the nonlinear diffusion equation:

$$u_t = \Delta\, u^m, \quad \text{in } Q_T = \Omega \times (0,T] \tag{5}$$

$$u(x,0) = u_0(x), \quad \text{in } \Omega \tag{6}$$

$$u(x,t) = 0, \quad \text{on } S_T = \partial\Omega \times (0,T] \tag{7}$$

where $m > 1$, $\Omega \subset \mathbb{R}^N$, with $N \geq 1$, is a bounded domain with boundary $\partial\Omega \in C^{2+\alpha}$, $\alpha \in (0,1)$, $u_0(x) \geq 0$;

Assuming temporarily that u is a smooth solution, multiply (5) by a smooth test function (whose properties are to be determined) and integrate over Q_T to find

$$0 = \iint_{Q_T} (u_t - \Delta\, u^m)\, \phi \, dx\, dt$$

Integrating by parts with respect to x, t, it follows that

$$0 = \iint_{Q_T} [\nabla u^m \cdot \nabla \phi - u\phi_t]\; dx\, dt + \int_\Omega u(x,T)\phi(x,T)\, dx -$$
$$- \int_\Omega u_0(x)\phi(x,0)\, dx - \int_{S_T} \frac{\partial u^m}{\partial \nu} \phi \, dx\, dt$$

where ν is an exterior normal at the boundary point $x \in \partial\Omega$. To simplify further, choose $\phi\Big|_{t=T} = \phi\Big|_{S_T} = 0$ to derive

$$\iint_{Q_T} (\nabla u^m \cdot \nabla \phi - u\phi_t)\; dx\, dt = \int_\Omega u_0(x)\phi(x,0)\, dx \tag{8}$$

which holds for all $\phi \in C^1(\bar{Q}_T)$ with $\phi(x,T) = 0$, $\phi\Big|_{S_T} = 0$.

The integral identity clearly holds for any smooth or classical solution of (5)-(7). However, opposite is not necessarily true, and in particular ZKB solution satisfies the integral identity (8), but not the equation (5). Hence, in order to give sense to ZKB type solutions, it makes sense to weaken the smoothness requirements on the solution by replacing PDE (5) with the integral identity (8). To exploit this idea, let us formulate minimal requirements on u which are enough to hold (8). Hence, integral identity will dictate the relevant Sobolev space for solutions. First of all, we are going to understand integration in the Lebesgue's sense, and all the derivatives are understood in a weak sense.

For the integral identity to make sense, we will need u^m in the Sobolev space

$$L_2\Big(0, T : H_0^1(\Omega)\Big).$$

Let's unpack this notation a bit. Recall that $L_2(\Omega)$ is a Banach space of Lebesgue measurable functions u with finite norm

$$\|u\|_{L_2(\Omega)} = \Big(\int_\Omega |u|^2 dx\Big)^{1/2}$$

It is an Hilbert space with inner product

$$(u, v) = \int_\Omega uv dx$$

and

$$H^1(\Omega) = \{u : \Omega \to \mathbb{R},\ u, \nabla u \in L_2(\Omega)\}$$

is a Banach space with norm

$$\|u\|_{H^1(\Omega)} = \Big(\int_\Omega |u|^2 + \|\nabla u\|^2\Big)^{1/2}$$

Then $H_0^1(\Omega)$ is the linear subspace of elements of $H^1(\Omega)$ which are "zero" on the boundary of Ω. To be more rigorous, $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the $H^1(\Omega)$ norm.

Now, we can define the Sobolev space $L_2\Big(0,T:H_0^1(\Omega)\Big)$ comprising functions mapping time into Bnach spaces:

$$L_2\Big(0,T:H_0^1(\Omega)\Big)=\Big\{u=u(t):[0,T]\to H_0^1(\Omega)\Big\}$$

that is, points in this space are maps from the interval $[0,T]$ to the Banach space $H_0^1(\Omega)$. This space has a norm

$$\|u\|_{L_2\left(0,T:H_0^1(\Omega)\right)}=\left(\int_0^T\|u(\cdot,t)\|^2_{H^1(\Omega)}\,dt\right)^{1/2}=\left(\int_0^T\int_\Omega|u|^2+\|\nabla u\|^2\,dx\,dt\right)^{1/2}$$

Finally note that for the right-hand side of (8) to make sense, it is enough to assume $u_0\in L_1(\Omega)$, space of Lebesgue integrable functions with the norm

$$\|u\|_{L_1(\Omega)}=\int_\Omega|u|dx$$

We can now formulate the notion of the weak solution of the problem (5)–(7):

Definition 1. We say that a function $u=u(x,t)$ is a weak solution of the Dirichlet problem for the Nonlinear Diffusion Equation, (5)–(7) if

(1) $u\geq 0$
(2) $u^m\in L_2(0,T;H_0^1(\Omega))$,
(3) u satisfies the integral identity

$$\iint_{Q_T}(\nabla u^m\cdot\nabla\phi-u\phi_t)\,dx\,dt=\int_\Omega u_0(x)\phi(x,0)\,dx$$

for any $\phi\in C^1(\overline{Q}_T)$ satisfying $\phi(x,T)=\phi\Big|_{S_T}=0$.

This definition gives sense to ZKB solution, and one can check that u_* is a weak solution of the relevant Dirichlet problem (recall that ∇u_*^m is a continuous function). How to make sure that the ZKB solution is the physically relevant solution, and in particular, there is no smooth classical solution of the nonlinear diffusion equation for the same data? To justify this we need to prove the uniqueness of the weak solution.

2. Uniqueness

Consider two solutions u_1,u_2 in the sense of Definition 1. Their difference satisfies the integral identity

$$\iint_{Q_T}\left[\nabla\left(u_1^m-u_2^m\right)\cdot\nabla\phi-\left(u_1-u_2\right)\phi_t\right]\,dx\,dt=0,\quad\forall\phi\in C^1(\bar{Q}_T)\text{ s.t. }\phi(x,T)=\phi\Big|_{S_T}=0\tag{9}$$

Due to the density of the space of smooth functions $C^1(\bar{Q}_T$ in a Sobolev space $H^1(Q_T)$, this integral identity holds for all $\phi\in H^1(Q_T)$ with $\phi(x,T)=\phi\Big|S_T=0$. Set

$$\eta(x,t)=\begin{cases}\int_t^T u_1^m(x,s)-u_2^m(x,s)\,ds, & 0<t<T\\ 0, & t\geq T\end{cases}$$

We calculate

$$\eta_t = -\left(u_1^m - u_2^m\right) \in L_2(Q_T),$$

$$\nabla \eta = \int_t^T \left(\nabla u_1^m - \nabla u_2^m\right) \, ds \in L_2(Q_T)$$

so η is an appropriate test function. Substituting it to (9) gives

$$\begin{aligned} 0 &= \iint_{Q_T} \left[\nabla \left(u_1^m - u_2^m\right) \cdot \int_t^T \left(\nabla u_1^m - \nabla u_2^m\right) \, ds + (u_1 - u_2)\left(u_1^m - u_2^m\right) \right] dx\, dt \\ &= \iint_{Q_T} \left[-\frac{1}{2} \frac{d}{dt} \left(\int_t^T \left(\nabla u_1^m - \nabla u_2^m\right) \, ds \right)^2 + (u_1 - u_2)\left(u_1^m - u_2^m\right) \right] dx\, dt \\ &= \frac{1}{2} \int_\Omega \left(\int_0^T \left(\nabla u_1^m - \nabla u_2^m\right) \, ds \right)^2 dx + \iint_{Q_T} (u_1 - u_2)\left(u_1^m - u_2^m\right) \, dx\, dt \end{aligned}$$

Note that both terms on the right-hand side are nonnegative, and accordingly both must be equal to zero. Hence, integrands must be equal to zero almost everywhere (a.e.). It follows that $u_1 = u_2$ a.e. in Q_T. Thus we proved the following result:

Theorem 1. *There exists at most one solution of the Dirichlet problem if* $u_0 \in L_1(\Omega)$.

In particular, it follows

Corollary 2. *There is no classical solution for ZKB datum i.e. initial data of the form* $u_*(x, t_0)$ *for fixed* t_0.

References

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[2] J.L.Vazquez, *The Porous Medium Equation. Mathematical Theory.*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007. xxii+624 pp.

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