Optimal Control and Inverse Problems for PDEs. Inverse Stefan Problem - Part 3

Ugur G. Abdulla

Florida Institute of Technology

May 16, 2014

Inverse Stefan Problem

Inverse Stefan Problem (ISP): *Find the functions $u(x,t)$ and $s(t)$ and the boundary heat flux $g(t)$ satisfying conditions*

$$(a(x,t)u_x)_x + b(x,t)u_x + c(x,t)u - u_t = f(x,t), \quad \text{for } (x,t) \in \Omega$$
$$u(x,0) = \phi(x), \qquad 0 \le x \le s(0) = s_0$$
$$a(0,t)u_x(0,t) = g(t), \qquad 0 \le t \le T$$
$$a(s(t),t)u_x(s(t),t) + \gamma(s(t),t)s'(t) = \chi(s(t),t), \qquad 0 \le t \le T$$
$$u(s(t),t) = \mu(t), \qquad 0 \le t \le T$$
$$u(0,t) = \nu(t), \quad \text{for } 0 \le t \le T$$

Optimal Control Problem

$$\mathcal{J}(v) = \beta_0 \|u(0,t) - \nu(t)\|^2_{L_2[0,T]} + \beta_1 \|u(s(t),t) - \mu(t)\|^2_{L_2[0,T]} \qquad (1)$$

$$V_R = \Big\{ v = (s,g) \in W_2^2[0,T] \times W_2^1[0,T] : \delta \le s(t) \le l,$$
$$s(0) = s_0, \max(\ \|s\|_{W_2^2};\ \|g\|_{W_2^1}) \le R \Big\}$$

$$(a(x,t)u_x)_x + b(x,t)u_x + c(x,t)u - u_t = f(x,t), \quad \text{for } (x,t) \in \Omega \qquad (2)$$
$$u(x,0) = \phi(x), \qquad 0 \le x \le s(0) = s_0 \qquad (3)$$
$$a(0,t)u_x(0,t) = g(t), \qquad 0 \le t \le T \qquad (4)$$
$$a(s(t),t)u_x(s(t),t) + \gamma(s(t),t)s'(t) = \chi(s(t),t), \qquad 0 \le t \le T \qquad (5)$$
$$\Omega = \{(x,t) :\ 0 < x < s(t),\ 0 < t \le T\}$$

Discrete Optimal Control Problem

$$\omega_\tau = \{t_j = j \cdot \tau,\ j = 0, 1, \dots, n\}$$

$$V_R^n = \{[v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} :\ 0 < \delta \leq s_k \leq l,$$

$$\max(\|[s]_n\|^2_{w_2^2};\ \|[g]_n\|^2_{w_2^1}) \leq R^2\}$$

$$[s]_n = (s_0, s_1, ..., s_n) \in \mathbb{R}^{n+1},\ [g]_n = (g_0, g_1, ..., g_n) \in \mathbb{R}^{n+1}$$

$$\|[s]_n\|^2_{w_2^2} = \sum_{k=0}^{n-1} \tau s_k^2 + \sum_{k=1}^{n} \tau s_{\bar{t},k}^2 + \sum_{k=1}^{n-1} \tau s_{\bar{t}t,k}^2,\ \|[g]_n\|^2_{w_2^1} = \sum_{k=0}^{n-1} \tau g_k^2 + \sum_{k=1}^{n} \tau g_{\bar{t},k}^2.$$

$$s_{\bar{t},k} = \frac{s_k - s_{k-1}}{\tau},\ s_{t,k} = \frac{s_{k+1} - s_k}{\tau},\ s_{\bar{t}t,k} = \frac{s_{k+1} - 2s_k + s_{k-1}}{\tau^2}.$$

Discrete State Vector

Definition 1

$[u([v]_n)]_n = (u(x;0), u(x;1), ..., u(x;n))$ is called discrete state vector if

1. $u(x;0) = \phi(x) \in W_2^1[0, s_0]$;
2. $u(x;k) \in W_2^1[0, s_k]$ satisfy the integral identity

$$\int_0^{s_k} \Big(a_k(x) \frac{du(x;k)}{dx} \eta'(x) - b_k \frac{du(x;k)}{dx} \eta(x) - c_k(x) u(x;k) \eta(x) + f_k(x) \eta + u_{\overline{t}}(x;k) \eta(x) \Big) dx + \Big((\gamma_{s^n} (s^n)')^k - \chi_{s^n}^k \Big) \eta(s_k) + g_k^n \eta(0) = 0,$$

$$\forall \eta \in W_2^1[0, s_k] \qquad (6)$$

3. $u(x;k) \in W_2^1[0, s_k]$ iteratively continued to $[0, l]$ as

$$u(x;k) = u(2^n s_k - x; k), \;\; 2^{n-1} s_k \leq x \leq 2^n s_k, n = \overline{1, n_k},$$

$$n_k \leq N = 1 + \log_2 \left[\frac{l}{\delta} \right] \qquad (7)$$

Discrete Optimal Control Problem

$$\mathcal{I}_n([v]_n) = \beta_0 \tau \sum_{k=1}^{n} \Big(u(0;k) - \nu_k\Big)^2 + \beta_1 \tau \sum_{k=1}^{n} \Big(u(s_k;k) - \mu_k\Big)^2 \quad (8)$$

$$V_R^n = \{[v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} : \ 0 < \delta \le s_k \le l,$$
$$\max(\|[s]_n\|^2_{w_2^2}; \ \|[g]_n\|^2_{w_2^1}) \le R^2\}$$

$[u([v]_n)]_n = (u(x;0), u(x;1), ..., u(x;n))$ be a discrete state vector.
Formulated discrete optimal control problem will be called Problem $\mathcal{I}_n$.

$$u^\tau(x,t) = u(x;k), \quad \text{if } t_{k-1} < t \le t_k, \ 0 \le x \le l, \ k = \overline{1,n},$$

$$\hat{u}^\tau(x,t) = u(x;k-1) + u_{\bar{t}}(x;k)(t - t_{k-1}), \quad \text{if } t_{k-1} < t \le t_k, \ 0 \le x \le l, \ k = \overline{1,n}$$

$$\hat{u}^\tau(x,t) = u(x;n), \quad \text{if } t \ge T, \ 0 \le x \le l.$$

$$u^\tau \in V_2(D), \quad \hat{u}^\tau \in W_2^{1,1}(D)$$

Existence of the Optimal Control

$$D = \{(x,t) : \ 0 < x < l, \ 0 < t \leq T\}$$

$$a, b, c \in L_\infty(D), \ f \in L_2(D),$$

$$\phi \in W_2^1[0, s_0], \ \gamma, \chi \in W_2^{1,1}(D), \ \mu, \nu \in L_2[0,T],$$

$$\frac{\partial a}{\partial x} \in L_\infty(D), \quad \int_0^T \operatorname*{ess\,sup}_{0 \leq x \leq l} \left| \frac{\partial a}{\partial t} \right| dt < +\infty. \tag{9}$$

Theorem 2

The Problem $\mathcal{I}$ has a solution, i.e.

$$V_* = \{v \in V_R : \mathcal{J}(v) = \mathcal{J}_* \equiv \inf_{v \in V_R} \mathcal{J}(v)\} \neq \emptyset$$

Convergence Theorem

Theorem 3

Sequence of discrete optimal control problems $\mathcal{I}_n$ *approximates the optimal control problem* $\mathcal{I}$ *with respect to functional, i.e.*

$$\lim_{n\to+\infty} I_{n_*} = J_*, \tag{10}$$

where

$$I_{n_*} = \inf_{V_R^n} I_n([v]_n), \ n = 1, 2, ...$$

If $[v]_{n_\epsilon} \in V_R^n$ *is chosen such that*

$$I_{n_*} \leq I_n([v]_{n_\epsilon}) \leq I_{n_*} + \epsilon_n, \ \epsilon_n \downarrow 0,$$

then the sequence $v_n = (s_n, g_n) = \mathcal{P}_n([v]_{n_\epsilon})$ *converges to some element* $v_* = (s_*, g_*) \in V_*$ *weakly in* $W_2^2[0,T] \times W_2^1[0,T]$*, and strongly in* $W_2^1[0,T] \times L_2[0,T]$*. In particular* s_n *converges to* s_* *uniformly on* $[0,T]$*. Moreover, piecewise linear interpolation* $\hat{u}^\tau$ *of the discrete state vector* $[u[v]_{n_\epsilon}]_n$ *converges to the solution* $u(x,t;v_*) \in W_2^{1,1}(\Omega_*)$ *of the Neumann problem (2)-(5) weakly in* $W_2^{1,1}(\Omega_*)$*.*

First Energy Estimate and its Consequences

Theorem 4

For all sufficiently small τ discrete state vector $[u([v]_n)]_n$ satisfies:

$$\max_{0\le k\le n}\int_0^l u^2(x;k)\,dx+\tau\sum_{k=1}^n\int_0^l\Big|\frac{du(x;k)}{dx}\Big|^2dx\le$$

$$C\Big(\|\phi\|^2_{L_2(0,s_0)}+\|g^n\|^2_{L_2(0,T)}+\|f\|^2_{L_2(D)}+\|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)}$$

$$+\|\chi(s^n(t),t)\|^2_{L_2(0,T)}+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}}u^2(x;k)dx\Big),\qquad (11)$$

$$\max_{0\le k\le n}\int_0^l u^2(x;k)\,dx+\tau\sum_{k=0}^n\int_0^l\Big|\frac{du(x;k)}{dx}\Big|^2dx+\tau^2\sum_{k=1}^n\int_0^l u_{\bar t}^2(x;k)dx\le$$

$$C\Big(\|\phi\|^2_{W_2^1(0,s_0)}+\|g^n\|^2_{L_2(0,T)}+\|f\|^2_{L_2(D)}+\|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)}$$

$$+\|\chi(s^n(t),t)\|^2_{L_2(0,T)}+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}}u^2(x;k)dx\Big),\qquad (12)$$

Weak Compactness in $V_2^{1,0}$

Theorem 5

*Let $[v]_n \in V_R^n, n = 1, 2, ...$ be a sequence of discrete controls and the sequence $\{\mathcal{P}_n([v]_n\}$ converges strongly in $W_2^1[0,T] \times L_2[0,T]$ to $v = (s, g)$. Then the sequence $\{u^\tau\}$ converges as $\tau \to 0$ weakly in $W_2^{1,0}(\Omega)$ to weak solution $u \in V_2^{1,0}(\Omega)$ of the problem (2)-(5), i.e. to the solution of the integral identity (**??**). Moreover, u satisfies the energy estimate*

$$\|u\|^2_{V_2^{1,0}(D)} \leq C\Big(\|\phi\|^2_{L_2(0,s_0)} + \|g\|^2_{L_2(0,T)} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,0}(D)} + \|\chi\|^2_{W_2^{1,0}(D)}\Big) \tag{13}$$

Corollary 6

For arbitrary $v = (s, g) \in V_R$ there exists a weak solution $u \in V_2^{1,0}(\Omega)$ of the problem (2)-(5) which satisfy the energy estimate (6):

$$\|u\|^2_{V_2^{1,0}(D)} \leq C\Big(\|\phi\|^2_{L_2(0,s_0)} + \|g\|^2_{L_2(0,T)} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,0}(D)} + \|\chi\|^2_{W_2^{1,0}(D)}\Big)$$

Second Energy Estimate and its Consequences

Theorem 7

For all sufficiently small τ discrete state vector $[u([v]_n)]_n$ satisfies the following stability estimation:

$$\max_{1\le k\le n}\int_0^l \left|\frac{du(x;k)}{dx}\right|^2 dx + \tau\sum_{k=1}^{n}\int_0^l u_{\bar{t}}^2(x;k)\,dx \le C\Bigg[\|\phi\|^2_{W_2^1[0,l]} +$$
$$\|g^n\|^2_{W_2^{\frac14}[0,T]} + \|\gamma(s^n(t),t)(s^n)'(t)\|^2_{W_2^{\frac14}[0,T]} + \|\chi(s^n(t),t)\|^2_{W_2^{\frac14}[0,T]} +$$
$$+\|f\|^2_{L_2(D)} + \sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}} u^2(x;k)dx\Bigg] \qquad (14)$$

$W_2^{\frac14}[0,T]$ – Banach space of all elements of $L_2[0,T]$ with finite norm

$$\|u\|_{W_2^{\frac14}[0,T]} = \left(\|u\|^2_{L_2[0,T]} + \int_0^T dt\int_0^T \frac{|u(t)-u(\tau)|^2}{|t-\tau|^{\frac32}}d\tau\right)^{\frac12}$$

Weak Compactness in $W_2^{1,1}$

Theorem 8

Let $[v]_n \in V_R^n, n = 1, 2, ...$ be a sequence of discrete controls and the sequence $\{\mathcal{P}_n([v]_n\}$ converges weakly in $W_2^2[0,T] \times W_2^1[0,T]$ to $v = (s, g)$. Then the sequence $\{\hat{u}^\tau\}$ converges as $\tau \to 0$ weakly in $W_2^{1,1}(\Omega)$ to weak solution $u \in W_2^{1,1}(\Omega)$ of the problem (2)-(5). Moreover, u satisfies the energy estimate

$$\|u\|^2_{W_2^{1,1}(D)} \leq C\Big(\|\phi\|^2_{W_2^1(0,s_0)} + \|g\|^2_{W_2^{\frac{1}{4}}} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,1}(D)} + \|\chi\|^2_{W_2^{1,1}(D)}\Big) \tag{15}$$

Corollary 9

For arbitrary $v = (s, g) \in V_R$ there exists a weak solution $u \in W_2^{1,1}(\Omega)$ of the problem (2)-(5) which satisfy the energy estimate

$$\|u\|^2_{W_2^{1,1}(D)} \leq C\Big(\|\phi\|^2_{W_2^1(0,s_0)} + \|g\|^2_{W_2^{\frac{1}{4}}} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,1}(D)} + \|\chi\|^2_{W_2^{1,1}(D)}\Big)$$

Proof of Convergence Theorem

Lemma 10

Let $\mathcal{J}_*(\pm\epsilon) = \inf_{V_{R\pm\epsilon}} \mathcal{J}(v)$, $\epsilon > 0$. *Then*

$$\lim_{\epsilon\to 0} \mathcal{J}_*(\epsilon) = \mathcal{J}_* = \lim_{\epsilon\to 0} \mathcal{J}_*(-\epsilon) \tag{16}$$

Lemma 11

For arbitrary $v = (s, g) \in V_R$,

$$\lim_{n\to\infty} \mathcal{I}_n(\mathcal{Q}_n(v)) = \mathcal{J}(v) \tag{17}$$

Lemma 12

For arbitrary $[v]_n \in V_R^n$

$$\lim_{n\to\infty} \Big(\mathcal{J}(\mathcal{P}_n([v]_n)) - \mathcal{I}_n([v]_n)\Big) = 0 \tag{18}$$

Proof of Convergence Theorem

Proof of Lemma 12

Let $[v]_n \in V_R^n$ and $v^n = (s^n, g^n) = \mathcal{P}_n([v]_n)$. From Lemma 2.3 it follows that the sequence $\{\mathcal{P}_n([v]_n\}$ is weakly precompact in $W_2^2[0,T] \times W_2^1[0,T]$.

Let $[v]_n \in V_R^n$ and $v^n = (s^n, g^n) = \mathcal{P}_n([v]_n)$. From Lemma 2.3 it follows that the sequence $\{\mathcal{P}_n([v]_n\}$ is weakly precompact in $W_2^2[0,T] \times W_2^1[0,T]$. Assume that the whole sequence converges to $\tilde{v} = (\tilde{s}, \tilde{g})$ weakly in $W_2^2[0,T] \times W_2^1[0,T]$.

Proof of Convergence Theorem

Proof of Lemma 12

Let $[v]_n \in V_R^n$ and $v^n = (s^n, g^n) = \mathcal{P}_n([v]_n)$. From Lemma 2.3 it follows that the sequence $\{\mathcal{P}_n([v]_n\}$ is weakly precompact in $W_2^2[0,T] \times W_2^1[0,T]$. Assume that the whole sequence converges to $\tilde{v} = (\tilde{s}, \tilde{g})$ weakly in $W_2^2[0,T] \times W_2^1[0,T]$. This implies the strong convergence in $W_2^1[0,T] \times L_2[0,T]$; $\tilde{v} \in V_R$.

Let $[v]_n \in V_R^n$ and $v^n = (s^n, g^n) = \mathcal{P}_n([v]_n)$. From Lemma 2.3 it follows that the sequence $\{\mathcal{P}_n([v]_n\}$ is weakly precompact in $W_2^2[0,T] \times W_2^1[0,T]$. Assume that the whole sequence converges to $\tilde{v} = (\tilde{s}, \tilde{g})$ weakly in $W_2^2[0,T] \times W_2^1[0,T]$. This implies the strong convergence in $W_2^1[0,T] \times L_2[0,T]$; $\tilde{v} \in V_R$. Let $\mathcal{Q}_n(\tilde{v}) = [\tilde{v}]_n$.

Let $[v]_n \in V_R^n$ and $v^n = (s^n, g^n) = \mathcal{P}_n([v]_n)$. From Lemma 2.3 it follows that the sequence $\{\mathcal{P}_n([v]_n\}$ is weakly precompact in $W_2^2[0,T] \times W_2^1[0,T]$. Assume that the whole sequence converges to $\tilde{v} = (\tilde{s}, \tilde{g})$ weakly in $W_2^2[0,T] \times W_2^1[0,T]$. This implies the strong convergence in $W_2^1[0,T] \times L_2[0,T]$; $\tilde{v} \in V_R$. Let $\mathcal{Q}_n(\tilde{v}) = [\tilde{v}]_n$.

$$\mathcal{I}_n\big([v]_n\big) - \mathcal{J}(v^n) = \mathcal{I}_n\big([v]_n\big) - \mathcal{I}_n\big([\tilde{v}]_n\big) + \mathcal{I}_n\big([\tilde{v}]_n\big) - \mathcal{J}(\tilde{v}) + \mathcal{J}(\tilde{v}) - \mathcal{J}(v^n) \tag{19}$$

Proof of Convergence Theorem

Proof of Lemma 12

Let $[v]_n \in V_R^n$ and $v^n = (s^n, g^n) = \mathcal{P}_n([v]_n)$. From Lemma 2.3 it follows that the sequence $\{\mathcal{P}_n([v]_n\}$ is weakly precompact in $W_2^2[0,T] \times W_2^1[0,T]$. Assume that the whole sequence converges to $\tilde{v} = (\tilde{s}, \tilde{g})$ weakly in $W_2^2[0,T] \times W_2^1[0,T]$. This implies the strong convergence in $W_2^1[0,T] \times L_2[0,T]$; $\tilde{v} \in V_R$. Let $\mathcal{Q}_n(\tilde{v}) = [\tilde{v}]_n$.

$$\mathcal{I}_n\big([v]_n\big) - \mathcal{J}(v^n) = \mathcal{I}_n\big([v]_n\big) - \mathcal{I}_n\big([\tilde{v}]_n\big) + \mathcal{I}_n\big([\tilde{v}]_n\big) - \mathcal{J}(\tilde{v}) + \mathcal{J}(\tilde{v}) - \mathcal{J}(v^n) \tag{19}$$

We proved the weak continuity of the functional $\mathcal{J}(v)$, i.e.

$$\lim_{n\to\infty} \left(\mathcal{J}(\tilde{v}) - \mathcal{J}(v^n)\right) = 0.$$

Let $[v]_n \in V_R^n$ and $v^n = (s^n, g^n) = \mathcal{P}_n([v]_n)$. From Lemma 2.3 it follows that the sequence $\{\mathcal{P}_n([v]_n\}$ is weakly precompact in $W_2^2[0,T] \times W_2^1[0,T]$. Assume that the whole sequence converges to $\tilde{v} = (\tilde{s}, \tilde{g})$ weakly in $W_2^2[0,T] \times W_2^1[0,T]$. This implies the strong convergence in $W_2^1[0,T] \times L_2[0,T]$; $\tilde{v} \in V_R$. Let $\mathcal{Q}_n(\tilde{v}) = [\tilde{v}]_n$.

$$\mathcal{I}_n\big([v]_n\big) - \mathcal{J}(v^n) = \mathcal{I}_n\big([v]_n\big) - \mathcal{I}_n\big([\tilde{v}]_n\big) + \mathcal{I}_n\big([\tilde{v}]_n\big) - \mathcal{J}(\tilde{v}) + \mathcal{J}(\tilde{v}) - \mathcal{J}(v^n) \tag{19}$$

We proved the weak continuity of the functional $\mathcal{J}(v)$, i.e.

$$\lim_{n\to\infty} \left(\mathcal{J}(\tilde{v}) - \mathcal{J}(v^n)\right) = 0.$$

From Lemma 11 it follows that

$$\lim_{n\to\infty} \Big(\mathcal{I}_n\big([\tilde{v}]_n\big) - \mathcal{J}(\tilde{v})\Big) = 0.$$

Let $[v]_n \in V_R^n$ and $v^n = (s^n, g^n) = \mathcal{P}_n([v]_n)$. From Lemma 2.3 it follows that the sequence $\{\mathcal{P}_n([v]_n\}$ is weakly precompact in $W_2^2[0,T] \times W_2^1[0,T]$. Assume that the whole sequence converges to $\tilde{v} = (\tilde{s}, \tilde{g})$ weakly in $W_2^2[0,T] \times W_2^1[0,T]$. This implies the strong convergence in $W_2^1[0,T] \times L_2[0,T]$; $\tilde{v} \in V_R$. Let $\mathcal{Q}_n(\tilde{v}) = [\tilde{v}]_n$.

$$\mathcal{I}_n\big([v]_n\big) - \mathcal{J}(v^n) = \mathcal{I}_n\big([v]_n\big) - \mathcal{I}_n\big([\tilde{v}]_n\big) + \mathcal{I}_n\big([\tilde{v}]_n\big) - \mathcal{J}(\tilde{v}) + \mathcal{J}(\tilde{v}) - \mathcal{J}(v^n) \tag{19}$$

We proved the weak continuity of the functional $\mathcal{J}(v)$, i.e.

$$\lim_{n\to\infty} \left(\mathcal{J}(\tilde{v}) - \mathcal{J}(v^n)\right) = 0.$$

From Lemma 11 it follows that

$$\lim_{n\to\infty} \Big(\mathcal{I}_n\big([\tilde{v}]_n\big) - \mathcal{J}(\tilde{v})\Big) = 0.$$

Hence, we only need to prove that

$$\lim_{n\to\infty} \Big(\mathcal{I}_n\big([v]_n\big) - \mathcal{I}_n\big([\tilde{v}]_n\big)\Big) = 0 \tag{20}$$

Proof of Convergence Theorem

Proof of Lemma 12, cont'd

$$[u([v]_n)]_n = (u_n(x;0), u_n(x;1), ..., u_n(x;n)),$$
$$[u([\tilde{v}]_n)]_n = (\tilde{u}(x;0), \tilde{u}(x;1), ..., \tilde{u}(x;n))$$
$$s_k = s^n(t_k), \tilde{s}_k = \tilde{s}(t_k),\ \Delta u(x;k) = u_n(x;k) - \tilde{u}(x;k)$$

$$[u([v]_n)]_n = (u_n(x;0), u_n(x;1), ..., u_n(x;n)),$$
$$[u([\tilde{v}]_n)]_n = (\tilde{u}(x;0), \tilde{u}(x;1), ..., \tilde{u}(x;n))$$
$$s_k = s^n(t_k), \tilde{s}_k = \tilde{s}(t_k),\ \Delta u(x;k) = u_n(x;k) - \tilde{u}(x;k)$$

$$\mathcal{I}_n([v]_n) - \mathcal{I}_n([\tilde{v}]_n) = \beta_0 \sum_{k=1}^{n} \tau \left(u_n(0;k) - f_0^k\right)^2 + \beta_1 \sum_{k=1}^{n} \tau \left(u_n(s_k;k) - f_1^k\right)^2$$
$$-\beta_0 \sum_{k=1}^{n} \tau \left(\tilde{u}(0;k) - f_0^k\right)^2 - \beta_1 \sum_{k=1}^{n} \tau \left(\tilde{u}(\tilde{s}_k;k) - f_1^k\right)^2 = \beta_0 \sum_{k=1}^{n} \tau \left(\Delta u(0;k)\right)^2 +$$
$$+2\beta_0 \sum_{k=1}^{n} \tau \Delta u(0;k) \left(\tilde{u}(0;k) - f_0^k\right) + \beta_1 \sum_{k=1}^{n} \tau \left(\Delta u(s_k;k) + \tilde{u}(s_k;k) - \tilde{u}(\tilde{s}_k;k)\right)^2$$
$$+2\beta_1 \sum_{k=1}^{n} \tau \left(\Delta u(s_k;k) + \tilde{u}(s_k;k) - \tilde{u}(\tilde{s}_k;k)\right) \left(\tilde{u}(\tilde{s}_k;k) - f_1^k\right) \qquad (21)$$

Proof of Convergence Theorem

Proof of Lemma 12, cont'd

$$\sum_{k=1}^{n} \tau \left(\tilde{u}(s_k;k) - \tilde{u}(\tilde{s}_k;k)\right)^2 = \sum_{k=1}^{n} \tau \left| \int_{s_k}^{\tilde{s}_k} \frac{d\tilde{u}(x;k)}{dx} dx \right|^2$$

$$\leq \max_{1\leq k\leq n} |s_k - \tilde{s}_k| \sum_{k=1}^{n} \tau \left| \int_{s_k}^{\tilde{s}_k} \left| \frac{\partial \tilde{u}(x;k)}{\partial x} \right|^2 dx \right| \to 0, \quad \text{as } n \to +\infty. \quad (22)$$

Proof of Convergence Theorem

Proof of Lemma 12, cont'd

$$\sum_{k=1}^{n} \tau \left(\tilde{u}(s_k;k) - \tilde{u}(\tilde{s}_k;k)\right)^2 = \sum_{k=1}^{n} \tau \left| \int_{s_k}^{\tilde{s}_k} \frac{d\tilde{u}(x;k)}{dx} dx \right|^2$$

$$\leq \max_{1 \leq k \leq n} |s_k - \tilde{s}_k| \sum_{k=1}^{n} \tau \left| \int_{s_k}^{\tilde{s}_k} \left| \frac{\partial \tilde{u}(x;k)}{\partial x} \right|^2 dx \right| \to 0, \quad \text{as } n \to +\infty. \quad (22)$$

It is enough to prove that

$$R = \sum_{k=1}^{n} \tau \left[\left(\Delta u(0;k)\right)^2 + \left(\Delta u(s_k;k)\right)^2 \right] \to 0 \text{ as } n \to +\infty \qquad (23)$$

$$\sum_{k=1}^{n} \tau \left(\tilde{u}(s_k;k) - \tilde{u}(\tilde{s}_k;k) \right)^2 = \sum_{k=1}^{n} \tau \left| \int_{s_k}^{\tilde{s}_k} \frac{d\tilde{u}(x;k)}{dx} dx \right|^2$$

$$\leq \max_{1 \leq k \leq n} |s_k - \tilde{s}_k| \sum_{k=1}^{n} \tau \left| \int_{s_k}^{\tilde{s}_k} \left| \frac{\partial \tilde{u}(x;k)}{\partial x} \right|^2 dx \right| \to 0, \quad \text{as } n \to +\infty. \quad (22)$$

It is enough to prove that

$$R = \sum_{k=1}^{n} \tau \left[\left(\Delta u(0;k) \right)^2 + \left(\Delta u(s_k;k) \right)^2 \right] \to 0 \text{ as } n \to +\infty \qquad (23)$$

By the Morrey inequality

$$R \leq C \sum_{k=1}^{n} \tau \left[\int_0^{s_k} \left| \Delta u(x;k) \right|^2 dx + \int_0^{s_k} \left| \frac{d\Delta u(x;k)}{dx} \right|^2 dx \right] \qquad (24)$$

where C is independent of n.

Proof of Convergence Theorem

Proof of Lemma 12, cont'd

Let us subtract integral identities for $u_n(x;k)$ and $\tilde{u}(x;k)$, by assuming that the fixed test function η belongs to $W_2^1[0,l]$.

Let us subtract integral identities for $u_n(x;k)$ and $\tilde{u}(x;k)$, by assuming that the fixed test function η belongs to $W_2^1[0,l]$.

$$\int_0^{s_k} \left(a_k(x)\frac{d\Delta u}{dx}\frac{d\eta}{dx} - b_k(x)\frac{d\Delta u}{dx}\eta(x) - c_k(x)\Delta u\eta + \Delta u_{\bar{t}}\eta \right) dx -$$
$$\chi^k_{\tilde{s}}\left[\eta(s_k) - \eta(\tilde{s}_k)\right] + \int_{s_k}^{\tilde{s}_k} \Big(a_k(x)\frac{d\tilde{u}}{dx}\frac{d\eta}{dx} - b_k(x)\frac{d\tilde{u}}{dx}\eta - c_k(x)\tilde{u}\eta +$$
$$f_k(x)\eta + \tilde{u}_{\bar{t}}\eta\Big) dx + (g_k^n - \tilde{g}_k^n)\eta(0) + \left[(\gamma_{s^n}(s^n)')^k - (\gamma_{\tilde{s}}\tilde{s}')^k\right]\eta(s_k) +$$
$$(\gamma_{\tilde{s}}\tilde{s}')^k\left[\eta(s_k) - \eta(\tilde{s}_k)\right] - \left[\chi^k_{s^n} - \chi^k_{\tilde{s}}\right]\eta(s_k) = 0 \qquad (25)$$

Our goal now is to derive from (25) that the right-hand side of (24) converges to zero as $n \to +\infty$. The proof goes along the same lines as the derivation of the first energy estimate:

Our goal now is to derive from (25) that the right-hand side of (24) converges to zero as $n \to +\infty$. The proof goes along the same lines as the derivation of the first energy estimate:

$$a_0\tau \int_0^{s_k} \left| \frac{d\Delta u(x;k)}{dx} \right|^2 dx + \int_0^{s_k} \Delta u^2(x;k)\, dx - \int_0^{s_k} \Delta u^2(s;k-1)\, dx +$$

$$\tau^2 \int_0^{s_k} \Delta u_{\bar{t}}^2(x;k) dx \leq C_1\tau \Big[\int_0^{s_k} \Delta u^2(x;k)\, dx + |g_k^n - \tilde{g}_k^n|^2 \Big] -$$

$$2\tau \int_{s_k}^{\tilde{s}_k} \Big[a_k(x) \frac{d\tilde{u}}{dx} \frac{d\Delta u(x;k)}{dx} - b_k(x) \frac{d\tilde{u}}{dx} \Delta u(x;k) - c_k(x)\tilde{u}\Delta u(x;k) + f_k(x) \times$$

$$\times \Delta u(x;k) + \tilde{u}_{\bar{t}} \Delta u(x;k) \Big] dx - 2 \int_{t_{k-1}}^{t_k} \Big(\gamma(s^n(t),t)(s^n)'(t) - \gamma(\tilde{s}(t),t)\tilde{s}'(t) \Big) dt \times$$

$$\times \Delta u(s_k;k) - 2 \int_{t_{k-1}}^{t_k} \int_{\tilde{s}_k}^{s_k} \gamma(\tilde{s}(t),t)\tilde{s}'(t) \frac{d\Delta u(x;k)}{dx}\, dx\, dt + 2 \int_{t_{k-1}}^{t_k} \Big(\chi(s^n(t),t) -$$

$$\chi(\tilde{s}(t),t) \Big) dt \Delta u(s_k;k) + 2 \int_{t_{k-1}}^{t_k} \int_{\tilde{s}_k}^{s_k} \chi(\tilde{s}(t),t) \frac{d\Delta u(x;k)}{dx}\, dx\, dt \qquad (26)$$

By applying the technique developed along with the proof of the first energy est. from (26) it follows that for all sufficiently small τ

$$\max_{1\leq k\leq n}\int_0^{s_k}\Delta u^2(x;k)\,dx \leq C\Big(\|g^n-\tilde{g}\|^2_{L_2[0,T]}+ \sum_{j=1}^{n-1}\mathbf{1}_+(s_{j+1}-s_j)\int_{s_j}^{s_{j+1}}\Delta u^2(x;j)dx+\sum_{j=1}^{n}|\mathcal{L}_j|\Big) \tag{27}$$

By applying the technique developed along with the proof of the first energy est. from (26) it follows that for all sufficiently small τ

$$\max_{1\leq k\leq n}\int_0^{s_k}\Delta u^2(x;k)\,dx \leq C\Big(\|g^n-\tilde{g}\|^2_{L_2[0,T]}+ \sum_{j=1}^{n-1}\mathbf{1}_+(s_{j+1}-s_j)\int_{s_j}^{s_{j+1}}\Delta u^2(x;j)dx+\sum_{j=1}^{n}|\mathcal{L}_j|\Big) \qquad (27)$$

$$\mathcal{L}_j=\tau\int_{s_j}^{\tilde{s}_j}\Big(a_j(x)\frac{d\tilde{u}}{dx}\frac{d\Delta u(x;j)}{dx}-b_j(x)\frac{d\tilde{u}}{dx}\Delta u(x;j)-c_j(x)\tilde{u}\Delta u(x;j)+ f_j(x)\Delta u(x;j)+\tilde{u}_{\bar{t}}\Delta u(x;j)\Big)\,dx+\int_{t_{j-1}}^{t_j}\left[\gamma(s^n(t),t)(s^n)'(t)-\gamma(\tilde{s}(t),t)\tilde{s}'(t)\right]dt \times\Delta u(s_j;j)+\int_{t_{j-1}}^{t_j}\int_{\tilde{s}_j}^{s_j}\gamma(\tilde{s}(t),t)\tilde{s}'(t)\frac{d\Delta u(x;j)}{dx}\,dx\,dt+\int_{t_{j-1}}^{t_j}\Big(\chi(s^n(t),t)- \chi(\tilde{s}(t),t)\Big)dt\Delta u(s_j;j)+\int_{t_{j-1}}^{t_j}\int_{\tilde{s}_j}^{s_j}\chi(\tilde{s}(t),t)\frac{d\Delta u(x;j)}{dx}\,dx\,dt \qquad (28)$$

Having (27) we perform summation in (26) with respect to k from 1 to n and derive

$$\max_{1\leq k\leq n}\int_0^{s_k}\Delta u^2(x;k)\,dx+\sum_{k=1}^n\tau\int_0^{s_k}\left|\frac{d\Delta u(x;k)}{dx}\right|^2dx+$$
$$\sum_{k=1}^n\tau^2\int_0^{s_k}\Delta u_{\bar{t}}^2(x;k)\,dx\leq C_1\Big(\|g^n-\tilde{g}\|^2_{L_2[0,T]}+$$
$$\sum_{j=1}^{n-1}\mathbf{1}_+(s_{j+1}-s_j)\int_{s_j}^{s_{j+1}}\Delta u^2(x;j)dx+\sum_{j=1}^n|\mathcal{L}_j|\Big) \qquad (29)$$

$$\Omega_n^1=\{0<t<T,0<x<\tilde{s}_1^n(t)\},\ \Omega_n=\bigcup_{k=1}^n\{t_{k-1}<t\leq t_k,0<x<s_k\}$$

Proof of Convergence Theorem

Proof of Lemma 12, cont'd

$$\|\Delta u^\tau\|^2_{V_2^{1,0}(\Omega_n)} = \max_{1\le k\le n}\int_0^{s_k}\Delta u^2(x;k)\,dx + \sum_{k=1}^n \tau\int_0^{s_k}\left|\frac{d\Delta u(x;k)}{dx}\right|^2 dx$$

$$\|\Delta u^\tau\|^2_{V_2^{1,0}(\Omega_n)} \le C_2\Big(\|g^n-\tilde g\|^2_{L_2[0,T]} + \|\Delta u^\tau\|^2_{V_2^{1,0}(\Omega_n^1-\Omega_n)} + \sum_{j=1}^n |\mathcal{L}_j|\Big) \tag{30}$$

Since

$$\|g^n-\tilde g\|_{L_2[0,T]}\to 0, \text{ as } \; n\to+\infty,$$

it is enough to prove that

$$\lim_{n\to+\infty}\sum_{j=1}^n |\mathcal{L}_j| = 0. \tag{31}$$

$$\sum_{j=1}^{n} \left| \tau \int_{s_j}^{\tilde{s}_j} a_j(x) \frac{d\tilde{u}(x;j)}{dx} \frac{d\Delta u(x;j)}{dx}\, dx \right| =$$

$$\sum_{j=1}^{n} \left| \int_{t_{j-1}}^{t_j} \int_{s_j}^{\tilde{s}_j} a(x,t) \frac{d\tilde{u}^\tau}{dx} \frac{d\Delta u^\tau}{dx}\, dx \right| \leq C \left\| \frac{\partial \tilde{u}^\tau}{\partial x} \right\|_{L_2(\tilde{\Delta})} \left\| \frac{\partial \Delta u^\tau}{\partial x} \right\|_{L_2(\tilde{\Delta})} \tag{32}$$

$$\tilde{\Delta} = \bigcup_{j=1}^{n} \{(x,t) : t_{j-1} < t < t_j,\ \min(s_j, \tilde{s}_j) < x < \max(s_j, \tilde{s}_j)\}$$

Proof of Convergence Theorem

Proof of Lemma 12, cont'd

$$\sum_{j=1}^{n} \left| \tau \int_{s_j}^{\tilde{s}_j} a_j(x) \frac{d\tilde{u}(x;j)}{dx} \frac{d\Delta u(x;j)}{dx}\, dx \right| =$$

$$\sum_{j=1}^{n} \left| \int_{t_{j-1}}^{t_j} \int_{s_j}^{\tilde{s}_j} a(x,t) \frac{d\tilde{u}^\tau}{dx} \frac{d\Delta u^\tau}{dx}\, dx \right| \leq C \left\| \frac{\partial \tilde{u}^\tau}{\partial x} \right\|_{L_2(\tilde{\Delta})} \left\| \frac{\partial \Delta u^\tau}{\partial x} \right\|_{L_2(\tilde{\Delta})} \tag{32}$$

$$\tilde{\Delta} = \bigcup_{j=1}^{n} \{(x,t) : t_{j-1} < t < t_j,\ \ \min(s_j, \tilde{s}_j) < x < \max(s_j, \tilde{s}_j)\}$$

$$\sum_{j=1}^{n} \left| \tau \int_{s_j}^{\tilde{s}_j} \tilde{u}_{\bar{t}}(x;j) \Delta u(x;j)\, dx \right| \leq \left(\sum_{j=1}^{n} \tau \int_0^l \tilde{u}_{\bar{t}}^2(x;j)\, dx \right)^{\frac{1}{2}} \|\Delta u^\tau\|_{L_2(\tilde{\Delta})} \tag{33}$$

$$
\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left| \Big(\gamma(s^n(t),t)(s^n)'(t) - \gamma(\tilde{s}(t),t)\tilde{s}'(t) \Big) \Delta u(s_j, j) \right| dt \leq
$$

$$
\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left| \Big(\gamma(s^n(t),t)(s^n)'(t) - \gamma(\tilde{s}(t),t)\tilde{s}'(t) \Big) \Big(\int_{\tilde{s}(t)}^{s_j} \frac{\partial \Delta u^\tau(x,t)}{\partial x} dx +
$$

$$
\Delta u(\tilde{s}(t); j) \Big) \right| dt \leq \|\gamma(s^n(t),t) - \gamma(\tilde{s}(t),t)\|_{L_2[0,T]} \|(s^n)'\|_{C[0,T]} \|\Delta u^\tau(\tilde{s}(t),t)\|_{L_2}
$$

$$
+ \|(s^n)'(t) - \tilde{s}'(t)\|_{L_2[0,T]} \|\gamma(\tilde{s}(t),t)\|_{L_4[0,T]} \|\Delta u^\tau(\tilde{s}(t),t)\|_{L_4[0,T]} +
$$

$$
\sum_{j=1}^{n} \left(\int_{t_{j-1}}^{t_j} |\gamma(s^n(t),t)(s^n)'(t) - \gamma(\tilde{s}(t),t)\tilde{s}'(t)|^2 \, dt \right)^{\frac{1}{2}} \left(\int_{t_{j-1}}^{t_j} \left| \int_{\tilde{s}(t)}^{s_j} \frac{\partial \Delta u^\tau}{\partial x} \, dx \right|^2 \right)
$$

$$
\leq C \|\gamma(s^n(t),t) - \gamma(\tilde{s}(t),t)\|_{L_2[0,T]} \|(s^n)'\|_{W_2^1[0,T]} \|\Delta u^\tau\|_{W_2^{1,0}(D)} +
$$

$$
C \|(s^n)'(t) - \tilde{s}'(t)\|_{L_2[0,T]} \|\gamma\|_{W_2^{1,1}(D)} \|\Delta u^\tau\|_{V_2(D)} + \left\| \frac{\partial \Delta u^\tau}{\partial x} \right\|_{L_2(\tilde{\Delta})} \times
$$

$$
\|\gamma(s^n(t),t)(s^n)'(t) - \gamma(\tilde{s}(t),t)\tilde{s}'(t)\|_{L^2[0,T]} \Big(\max_{0 \leq t \leq T} |\tilde{s}(t) - s^n(t)| \Big)^{\frac{1}{2}} \tag{34}
$$

- U. G. Abdulla.
  On the Optimal Control of the Free Boundary Problems for the Second Order Parabolic Equations. I. Well-posedness and Convergence of the Method of Lines.
  Inverse Problems and Imaging, 7(2):307–340, May 2013.

Method of Finite Differences

Spatial grid

Given $[v]_n \in V_R^n$, let $(p_0, p_1, \cdots, p_n)$ be a permutation of $(0, 1, \cdots, n)$ according to order

$$s_{p_0} \leq s_{p_1} \leq \cdots \leq s_{p_n}$$

Method of Finite Differences

Spatial grid

Given $[v]_n \in V_R^n$, let $(p_0, p_1, \cdots, p_n)$ be a permutation of $(0, 1, \cdots, n)$ according to order

$$s_{p_0} \leq s_{p_1} \leq \cdots \leq s_{p_n}$$

$$\forall k \in \{0, 1, ..., n\} \; \exists! \; j_k \;\; \text{such that } s_k = s_{p_{j_k}}; \quad j := j_k \qquad (35)$$

Method of Finite Differences

Spatial grid

Given $[v]_n \in V_R^n$, let $(p_0, p_1, \cdots, p_n)$ be a permutation of $(0, 1, \cdots, n)$ according to order

$$s_{p_0} \leq s_{p_1} \leq \cdots \leq s_{p_n}$$

$$\forall k \in \{0, 1, ..., n\} \; \exists! \; j_k \; \text{ such that } s_k = s_{p_{j_k}}; \quad j := j_k \qquad (35)$$

$$\omega_{p_0} = \{x_i = i \cdot h, \; i = 0, 1, \ldots, m_0^{(n)}\} \quad \text{be a grid on } [0, s_{p_0}]; \; h = \frac{s_{p_0}}{m_0^{(n)}}$$

Given $[v]_n \in V_R^n$, let $(p_0, p_1, \cdots, p_n)$ be a permutation of $(0, 1, \cdots, n)$ according to order

$$s_{p_0} \leq s_{p_1} \leq \cdots \leq s_{p_n}$$

$$\forall k \in \{0, 1, ..., n\} \ \exists! \ j_k \ \text{ such that } s_k = s_{p_{j_k}}; \quad j := j_k \qquad (35)$$

$$\omega_{p_0} = \{x_i = i \cdot h, \ i = 0, 1, \ldots, m_0^{(n)}\} \quad \text{be a grid on } [0, s_{p_0}]; \ h = \frac{s_{p_0}}{m_0^{(n)}}$$

$$\omega_{p_k} = \{x_i : \ i = 0, 1, \cdots, m_k^{(n)} \quad \text{be a grid on } [0, s_{p_k}]; \ m_k^{(n)} \geq m_{k-1}^n\}$$

Given $[v]_n \in V_R^n$, let $(p_0, p_1, \cdots, p_n)$ be a permutation of $(0, 1, \cdots, n)$ according to order

$$s_{p_0} \leq s_{p_1} \leq \cdots \leq s_{p_n}$$

$$\forall k \in \{0, 1, ..., n\} \ \exists! \ j_k \ \text{ such that } s_k = s_{p_{j_k}}; \quad j := j_k \tag{35}$$

$$\omega_{p_0} = \{x_i = i \cdot h, \ i = 0, 1, \ldots, m_0^{(n)}\} \quad \text{be a grid on } [0, s_{p_0}]; \ h = \frac{s_{p_0}}{m_0^{(n)}}$$

$$\omega_{p_k} = \{x_i : \ i = 0, 1, \cdots, m_k^{(n)} \quad \text{be a grid on } [0, s_{p_k}]; \ m_k^{(n)} \geq m_{k-1}^n\}$$

if $s_{p_n} < l$, then introduce a grid on $[s_{p_n}, l] : \ \overline{\omega} = \{x_i : \ i = m_n^{(n)}, \cdots, N\}$

Given $[v]_n \in V_R^n$, let $(p_0, p_1, \cdots, p_n)$ be a permutation of $(0, 1, \cdots, n)$ according to order

$$s_{p_0} \leq s_{p_1} \leq \cdots \leq s_{p_n}$$

$$\forall k \in \{0, 1, ..., n\} \ \exists! \ j_k \ \text{ such that } s_k = s_{p_{j_k}}; \quad j := j_k \tag{35}$$

$$\omega_{p_0} = \{x_i = i \cdot h, \ i = 0, 1, \ldots, m_0^{(n)}\} \quad \text{be a grid on } [0, s_{p_0}]; \ h = \frac{s_{p_0}}{m_0^{(n)}}$$

$$\omega_{p_k} = \{x_i : \ i = 0, 1, \cdots, m_k^{(n)} \quad \text{be a grid on } [0, s_{p_k}]; \ m_k^{(n)} \geq m_{k-1}^n\}$$

if $s_{p_n} < l$, then introduce a grid on $[s_{p_n}, l] : \ \overline{\omega} = \{x_i : \ i = m_n^{(n)}, \cdots, N\}$

$$h_i = x_{i+1} - x_i, \ i = 0, 1, \cdots, N-1;$$

Given $[v]_n \in V_R^n$, let $(p_0, p_1, \cdots, p_n)$ be a permutation of $(0, 1, \cdots, n)$ according to order

$$s_{p_0} \leq s_{p_1} \leq \cdots \leq s_{p_n}$$

$$\forall k \in \{0, 1, ..., n\} \ \exists! \ j_k \ \text{ such that } s_k = s_{p_{j_k}}; \quad j := j_k \tag{35}$$

$$\omega_{p_0} = \{x_i = i \cdot h, \ i = 0, 1, \ldots, m_0^{(n)}\} \quad \text{be a grid on } [0, s_{p_0}]; \ h = \frac{s_{p_0}}{m_0^{(n)}}$$

$$\omega_{p_k} = \{x_i : \ i = 0, 1, \cdots, m_k^{(n)} \quad \text{be a grid on } [0, s_{p_k}]; \ m_k^{(n)} \geq m_{k-1}^n\}$$

if $s_{p_n} < l$, then introduce a grid on $[s_{p_n}, l] : \ \overline{\omega} = \{x_i : \ i = m_n^{(n)}, \cdots, N\}$

$$h_i = x_{i+1} - x_i, \ i = 0, 1, \cdots, N-1;$$

$$m_k^{(n)} \equiv m_k, \ m_k \to +\infty, \ \max_{0 \leq i \leq N-1} h_i = O(\sqrt{\tau}), \quad \text{as } n \to \infty. \tag{36}$$

Method of Finite Differences

$$d_k(x) = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} d(x,t)\,dt,\;\; h_k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} h(t)\,dt,$$

$$d_{ik} = \frac{1}{h_i\tau}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k} d(x,t)\,dt\,dx,$$

where $i = 0, 1, \cdots, N-1;\;\; k = 1, \cdots, n;$ d stands for any of the functions a, b, c, f, and h stands for any of the functions ν, μ, g or g^n.

$$d_k(x) = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} d(x,t)\,dt,\ \ h_k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} h(t)\,dt,$$

$$d_{ik} = \frac{1}{h_i\tau}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k} d(x,t)\,dt\,dx,$$

where $i = 0, 1, \cdots, N-1;\ k = 1, \cdots, n$; d stands for any of the functions a, b, c, f, and h stands for any of the functions ν, μ, g or g^n. Given $v = (s, g) \in V_R$ we define Steklov averages of traces

$$\chi_s^k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \chi(s(t), t)\,dt,\ \ (\gamma_s s')^k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \gamma(s(t), t)s'(t)\,dt. \qquad (37)$$

Given $[v]_n = ([s]_n, [g]_n) \in V_R^n$ we define Steklov averages $\chi_{s^n}^k$ and $(\gamma_{s^n}(s^n)')^k$ through (37) with s replaced by s^n.

Discrete State Vector

$$[u([v]_n)]_n = (u(0), u(1), ..., u(n)),\ u(k) \in \mathbb{R}^{N+1},\ k = 0, \cdots, n$$

is called a discrete state vector if

(1) $u_i(0) = \frac{1}{h_i} \int_{x_i}^{x_{i+1}} \phi(x)\, dx,\ i = 0, 1, \cdots, m_0;$

(2) $u_i(k), k = 1, ..., n; i = 0, ..., m_j$ satisfy system of LAEs:

$$\Big[a_{0k} + hb_{0k} - h^2 c_{0k} + \frac{h^2}{\tau}\Big] u_0(k) - \Big[a_{0k} + hb_{0k}\Big] u_1(k) =$$

$$\frac{h^2}{\tau} u_0(k-1) - h^2 f_{0k} - hg_k^n,$$

$$-a_{i-1,k} h_i u_{i-1}(k) + \Big[a_{i-1,k} h_i + a_{ik} h_{i-1} + b_{ik} h_i h_{i-1} - c_{ik} h_i^2 h_{i-1} +$$

$$\frac{h_i^2 h_{i-1}}{\tau}\Big] u_i(k) - \Big[a_{ik} h_{i-1} + b_{ik} h_i h_{i-1}\Big] u_{i+1}(k) =$$

$$-h_i^2 h_{i-1} f_{ik} + \frac{h_i^2 h_{i-1}}{\tau} u_i(k-1),\ i = 1, \cdots, m_j - 1$$

$$-a_{m_j-1,k} u_{m_j-1}(k) + a_{m_j-1,k} u_{m_j}(k) = -h_{m_j-1}\Big[(\gamma_{s^n}(s^n)')^k - \chi_{s^n}^k\Big]. \tag{38}$$

Discrete State Vector

(3) For arbitrary $k = 0, 1, ..., n$, the remaining components of $u(k) \in \mathbb{R}^{N+1}$ are calculated as

$$u_i(k) = \hat{u}(x_i; k),\ m_j \leq i \leq N$$

where $\hat{u}(x; k) \in W_2^1[0, l]$ is a piecewise linear interpolation of $\{u_i(k) :\ i = 0, \cdots, m_j\}$, that is to say

$$\hat{u}(x; k) = u_i(k) + \frac{u_{i+1}(k) - u_i(k)}{h_i}(x - x_i),\ x_i \leq x \leq x_{i+1}, i = 0, \cdots, m_j - 1,$$

iteratively continued to $[0, l]$ as

$$\hat{u}(x; k) = \hat{u}(2^n s_k - x; k),\ 2^{n-1} s_k \leq x \leq 2^n s_k, n = \overline{1, n_k},$$
$$n_k \leq n_* = 1 + \log_2 \left[\frac{l}{\delta}\right] \tag{39}$$

where $[r]$ means integer part of the real number r.

Discrete State Vector

derivation

$$\int_0^{s_k} \Big(a_k(x)\frac{du(x;k)}{dx}\eta'(x) - b_k\frac{du(x;k)}{dx}\eta(x) - c_k(x)u(x;k)\eta(x) + f_k(x)\eta$$
$$+u_{\bar{t}}(x;k)\eta(x)\Big)\,dx + \Big((\gamma_{s^n}(s^n)')^k - \chi_{s^n}^k\Big)\eta(s_k) + g_k^n\eta(0) = 0,$$
$$\forall \eta \in W_2^1[0, s_k] \tag{40}$$

$$\int_0^{s_k} \Big(a_k(x) \frac{du(x;k)}{dx} \eta'(x) - b_k \frac{du(x;k)}{dx} \eta(x) - c_k(x) u(x;k) \eta(x) + f_k(x)\eta$$
$$+ u_{\bar{t}}(x;k)\eta(x) \Big) \, dx + \Big((\gamma_{s^n}(s^n)')^k - \chi^k_{s^n} \Big) \eta(s_k) + g^n_k \eta(0) = 0,$$
$$\forall \eta \in W_2^1[0, s_k] \tag{40}$$

$$\sum_{i=0}^{m_j - 1} h_i \Big(a_{ik} u_{ix}(k) \eta_{ix}(k) - b_{ik} u_{ix}(k) \eta_i(k) - c_{ik} u_i(k) \eta_i(k) + f_{ik} \eta_i(k)$$
$$+ u_{i\bar{t}}(k) \eta_i(k) \Big) + \Big((\gamma_{s^n}(s^n)')^k - \chi^k_{s^n} \Big) \eta_{m_j}(k) + g^n_k \eta_0(k) = 0,$$
$$\forall \eta_i(k), i = 0, .., m_j \tag{41}$$

By using summation by parts, we have

$$\sum_{i=1}^{m_j-1} h_i a_{ik} u_{ix}(k)\eta_{ix}(k) = \sum_{i=0}^{m_j-1} a_{ik} u_{ix}(k)\eta_{i+1}(k) - \sum_{i=0}^{m_j-1} a_{ik} u_{ix}(k)\eta_i(k)$$

$$= \sum_{i=1}^{m_j} a_{i-1,k} u_{i-1,x}(k)\eta_i(k) - \sum_{i=0}^{m_j-1} a_{ik} u_{ix}(k)\eta_i(k) =$$

$$\sum_{i=1}^{m_j-1} \Big(a_{i-1,k} u_{i-1,x}(k) - a_{ik} u_{ix}(k)\Big)\eta_i(k) +$$

$$a_{m_j-1,k} u_{m_j-1,x}(k)\eta_{m_j}(k) - a_{0k} u_{0x}(k)\eta_0(k). \tag{42}$$

Taking into account in (41):

Discrete State Vector

derivation

$$\sum_{i=1}^{m_j-1} h_i\Big(\frac{1}{h_i}\Big(a_{i-1,k}u_{i-1,x}(k) - a_{ik}u_{ix}(k)\Big)$$

$$-b_{ik}u_{ix}(k) - c_{ik}u_i(k) + f_{ik} + u_{i\bar{t}}(k)\Big)\eta_i(k)+$$

$$\Big(a_{m_j-1,k}u_{m_j-1,x}(k) + \Big((\gamma_{s^n}(s^n)')^k - \chi^k_{s^n}\Big)\Big)\eta_{m_j}(k)+$$

$$\Big(-a_{0k}u_{0,x}(k) + h(-b_{0k}u_{0x}(k) - c_{0k}u_0(k) + f_{0k} + u_{0\bar{t}}(k))\Big)\eta_0(k)$$

$$+g_k^n\eta_0(k) = 0 \qquad (43)$$

Since $\eta_i(k)$'s are arbitrary, system of LAE from the definition of the discrete state vector follows.

Fully Discretized Optimal Control Problem

$$\mathcal{I}_n([v]_n) = \beta_0 \tau \sum_{k=1}^{n} \Big(u_0(k) - \nu_k\Big)^2 + \beta_1 \tau \sum_{k=1}^{n} \Big(u_{m_{j_k}}(k) - \mu_k\Big)^2 \to \min$$

on a control set

$$V_R^n = \{[v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} : \ 0 < \delta \leq s_k \leq l, \\ \max(\|[s]_n\|^2_{w_2^2};\ \|[g]_n\|^2_{w_2^1}) \leq R^2\}$$

Formulated discrete optimal control problem will be called Problem I_n.

$$u^\tau(x,t)=\hat{u}(x;k),\quad \text{if } t_{k-1}<t\le t_k,\ 0\le x\le l,\ k=\overline{0,n},$$

$$\hat{u}^\tau(x,t)=\hat{u}(x;k-1)+\hat{u}_{\bar{t}}(x;k)(t-t_{k-1}),\quad \text{if } t_{k-1}<t\le t_k,\ 0\le x\le l,\ k=\overline{1,n},$$

$$\hat{u}^\tau(x,t)=\hat{u}(x;n),\quad \text{if } t\ge T,\ 0\le x\le l.$$

$$\tilde{u}^\tau(x,t)=u_i(k),\quad \text{if } t_{k-1}<t\le t_k,\ x_i\le x<x_{i+1},\ k=\overline{1,n},\ i=\overline{0,N-1}.$$

$$u^\tau\in V_2^{1,0}(D),\quad \hat{u}^\tau\in W_2^{1,1}(D),\quad \tilde{u}^\tau\in L_2(D).$$

$$u_{ix}(k)=\frac{u_{i+1}(k)-u_i(k)}{h_i},\ u_{i\bar{t}}=\frac{u_i(k)-u_i(k-1)}{\tau},\quad \text{etc.}$$

Assumptions

$$D = \{(x,t): \ 0 < x < l, \ 0 < t \leq T\}$$

$$a, b, c \in L_\infty(D), \ f \in L_2(D),$$

$$\phi \in W_2^1[0, s_0], \ \gamma, \chi \in W_2^{1,1}(D), \ \mu, \nu \in L_2[0,T],$$

$$\frac{\partial a}{\partial x} \in L_\infty(D), \quad \int_0^T esssup_{0\leq x\leq l} \left| \frac{\partial a}{\partial t} \right| dt < +\infty. \tag{44}$$

$$V_* = \{v \in V_R : \ \mathcal{J}(v) = \mathcal{J}_* \equiv \inf_{v \in V_R} \mathcal{J}(v)\}$$

Convergence Result

Theorem 13

Sequence of discrete optimal control problems I_n *approximates the optimal control problem* I *with respect to functional, i.e.*

$$\lim_{n\to+\infty} \mathcal{I}_{n_*} = \mathcal{J}_*, \tag{45}$$

where

$$\mathcal{I}_{n_*} = \inf_{V_R^n} \mathcal{I}_n([v]_n), \ n = 1, 2, ...$$

If $[v]_{n_\epsilon} \in V_R^n$ *is chosen such that*

$$\mathcal{I}_{n_*} \leq \mathcal{I}_n([v]_{n_\epsilon}) \leq \mathcal{I}_{n_*} + \epsilon_n, \ \epsilon_n \downarrow 0,$$

then the sequence $v_n = (s_n, g_n) = \mathcal{P}_n([v]_{n_\epsilon})$ *converges to some element* $v_* = (s_*, g_*) \in V_*$ *weakly in* $W_2^2[0,T] \times W_2^1[0,T]$*, and strongly in* $W_2^1[0,T] \times L_2[0,T]$*. In particular* s_n *converges to* s_* *uniformly on* $[0,T]$*. Moreover, piecewise linear interpolation* $\hat{u}^\tau$ *of the discrete state vector* $[u[v]_{n_\epsilon}]_n$ *converges to the solution* $u(x,t;v_*) \in W_2^{1,1}(\Omega_*)$ *of the Neumann problem (2)-(5) weakly in* $W_2^{1,1}(\Omega_*)$*.*

- U. G. Abdulla.
  On the Optimal Control of the Free Boundary Problems for the Second Order Parabolic Equations. II. Convergence of the Method of Finite Differences.
  Inverse Problems and Imaging, submitted.

References

- U.G.Abdulla,On the Optimal Control of the Free Boundary Problems for the Second Order Parabolic Equations. I. Well-posedness and Convergence of the Method of Lines,*Inverse Problems and Imaging*, **7**,*2(2013),307-340*.
- U.G.Abdulla,On the Optimal Control of the Free Boundary Problems for the Second Order Parabolic Equations. II. Convergence of the Method of FiniteDifferences,*Inverse Problems and Imaging, submitted*.
- O.M.Alifanov,*Inverse Heat Transfer Problems*,Springer-Verlag Telos,1995.
- J.Baumeister, Zur optimal Steuerung von frien Randwertausgaben,*ZAMM*,**60**,*(1980),335-339*.
- O.V.Besov, V.P.Il'in and S.M.Nikol'skii, *Integral Representations of Functions and Embedding Theorems*, Winston & Sons, Washington, D.C.; John Wiley & Sons, 1978.

J.B.Bell, The Non-Characteristic Cauchy Problem for a Class of Equations with Time Dependence.I.Problem in One Space Dimension,*SIAM J.Math.Anal.*,**12**,*(1981),759-777*.

B.M.Budak and V.N.Vasil'eva, On the solution of the inverse Stefan problem,*Soviet Math.Dokl*,**13**,*(1972),811-815*.

B.M.Budak and V.N.Vasil'eva, The solution of the inverse Stefan problem,*USSR Comput.Maths.Math.Phys*,**13**,*(1973),130-151*.

B.M.Budak and V.N.Vasil'eva, On the solution of Stefan's converse problem II,*USSR Comput.Maths.Math.Phys*,**13**,*(1973),97-110*.

J.R.Cannon, A Cauchy Problem for the Heat Equation,*Ann.Math*,**66**,*(1964),155-166*.

J.R.Cannon and J.Douglas, The Cauchy Problem for the Heat Equation,*SIAM.J.Numer.Anal*,**4**,*(1967),317-336*.

A.Carasso, Determining Surface Temperatures from Interior Observations,*SIAM J.Appl.Math*,**42**,*(1982),558-574*.

R.E.Ewing, The Cauchy Problem for a Linear Parabolic Equation,*J.Math.Anal.Appl.*,**71**,*(1970),167-186*.

- R.E.Ewing and R.S.Falk, Numerical Approximation of a Cauchy Problem for a Parabolic Partial Differential Equation,*Math.Comput.*,**33**,*(1979),1125-1144*.
- A.Fasano and M.Primicerio,General Free Boundary Problems for Heat Equations,*J.Math.Anal.Appl.*,**57**,*(1977),694-723*.
- A.Friedman,*Variational Principles and Free Boundary Problems*,John Wiley,1982.
- N.L.Gol'dman, *Inverse Stefan problems*. Mathematics and its Applications, 412. Kluwer Academic Publishers Group, Dordrecht, 1997. viii+250 pp
- N.L.Gol'dman,Properties of Solutions of the Inverse Stefan Problem,*Differential Equations*,**39**,*(2003),66-72*.
- K.H.Hoffman and M.Niezgodka,*Control of Parabolic Systems Involving Free Boundarie*, Proc.of Int.Conf. on Free Boundary Problems,1981.

- K.H.Hoffman and J.Sprekels, Real Time Control of Free Boundary in a Two-Phase Stefan Problem,*Numer.Funct.Anal.and Optimiz*,**5**,*(1982),47-76*.
- K.H. Hoffman and J.Sprekels, On the Identification of Heat Conductivity and Latent Heat Conductivity ans Latent Heat in a One-Phase Stefan Problem,*Control and Cybernetics*,**15**,*(1986),37-51*.
- P.Jochum,The Inverse Stefan problem as a problem of nonlinear approximation theory,*Journal of Approximate Theorey*,**30**,*(1980),81-98*.
- P.Jochum,The numerical solution of the inverse Stefan problem,*Numerical Mathematics*,**34**,*)(1980),411-429*.
- P.Knabner, Stability Theorems for General Free Boundary Problem of the Stefan Type and Applications,*Meth.Ser.Numer.Meth.Verf.Math.Phys.*,**25**,*(1983),95-116*.
- O.A.Ladyzenskaya, V.A.Solonnikov, N.N.Ural'ceva,*Linear and Quasilinear Equations of the Parabolic Type*, Translations of Mathematical Monographs, Vol. 23, American mathematical Society,

- K.A.Lurye,*Optimal Control in Problems of Mathematical Physics*,Nauka,Moscow,1975.
- A.M.Meyrmanov, *The Stefan Problem*,Walter de Gruyter, 1992.
- M.Niezgodka,Control of Parabolic Systems with Free Boundaries-Application of Inverse Formulation,*Control and Cybernetics*,**8**,*(1979),213-225*.
- S.M.Nikol'skii, *Approximation of Functions of Several Variables and Imbedding Theorems*, Springer-Verlag, New York-Heidelberg, 1975.
- R.H. Nochetto and C.Verdi,The Combined Use of Nonlinear Chernoff Formula with a Regularization Procedure for Two-Phase Stefan Problems,*Numer.Funct.Anal. and Optimiz.*,**9**,*(1987-1988),1177-1192*.
- M.Primicero,The Occurence of Pathologies in Some Stefan-like Problems,*Numerical Treatment of Free Boundary-Value Problems,ISNM,(1982),233-244*.

C.Sagues,Simulation and Optimal Control of Free Boundary,*Numerical Treatment of Free Boundary-Value Problems,ISNM,***58***,270-287*.

B.Sherman,General One-Phase Stefan Problems and Free Boundary Problems for the Heat Equation with Cauchy Data Prescribed on the Free Boundary,*SIAM J. Appl. Math.,***20***,(1971),557-570*.

V.A.Solonnikov, A priori estimates for solutions of second-order equations of parabolic type, *Trudy Mat. Inst. Steklov.,* **70**, *(1964), 133-212*.

V.A.Solonnikov, On boundary value problems for linear parabolic systems of differential equations of general form, *Trudy Mat. Inst. Steklov.,* **83**, *(1965), 3-163*.

G.Talenti and S.Vessella, A Note on an Ill-Posed Problem for the Heat Equation,*J.Austral.Math.Soc., Ser.A,***32***,(1982),358-368*.

- F.P.Vasil'ev,The existence of a solution of a certain optimal Stefan problem, In *Comput.Methods and Programming*,XII(Russian),1969,110-114.
- F.P.Vasil'ev, *Methods for Solving Extremal Problems. Minimization problems in function spaces, regularization, approximation*(in Russian), Moscow, Nauka, 1981.
- A.D.Yurii, On an Optimal Stefan Problem,*Dokl.Akad.Nauk SSSR*,**251**,*(1980),1317-1321*.