

Optimal Control and Inverse Problems for PDEs. Inverse Stefan Problem - Part 3

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Inverse Stefan Problem (ISP): Find the functions $u(x, t)$ and $s(t)$ and the boundary heat flux $g(t)$ satisfying conditions

$$(a(x, t)u_x)_x + b(x, t)u_x + c(x, t)u - u_t = f(x, t), \quad \text{for } (x, t) \in \Omega$$

$$u(x, 0) = \phi(x), \quad 0 \leq x \leq s(0) = s_0$$

$$a(0, t)u_x(0, t) = g(t), \quad 0 \leq t \leq T$$

$$a(s(t), t)u_x(s(t), t) + \gamma(s(t), t)s'(t) = \chi(s(t), t), \quad 0 \leq t \leq T$$

$$u(s(t), t) = \mu(t), \quad 0 \leq t \leq T$$

$$u(0, t) = \nu(t), \quad \text{for } 0 \leq t \leq T$$

Optimal Control Problem

$$\mathcal{J}(v) = \beta_0 \|u(0, t) - \nu(t)\|_{L_2[0, T]}^2 + \beta_1 \|u(s(t), t) - \mu(t)\|_{L_2[0, T]}^2 \quad (1)$$

$$V_R = \left\{ v = (s, g) \in W_2^2[0, T] \times W_2^1[0, T] : \delta \leq s(t) \leq l, \right. \\ \left. s(0) = s_0, \max(\|s\|_{W_2^2}; \|g\|_{W_2^1}) \leq R \right\}$$

$$(a(x, t)u_x)_x + b(x, t)u_x + c(x, t)u - u_t = f(x, t), \quad \text{for } (x, t) \in \Omega \quad (2)$$

$$u(x, 0) = \phi(x), \quad 0 \leq x \leq s(0) = s_0 \quad (3)$$

$$a(0, t)u_x(0, t) = g(t), \quad 0 \leq t \leq T \quad (4)$$

$$a(s(t), t)u_x(s(t), t) + \gamma(s(t), t)s'(t) = \chi(s(t), t), \quad 0 \leq t \leq T \quad (5)$$

$$\Omega = \{(x, t) : 0 < x < s(t), 0 < t \leq T\}$$

Discrete Optimal Control Problem

$$\omega_\tau = \{t_j = j \cdot \tau, j = 0, 1, \dots, n\}$$

$$V_R^n = \{[v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} : 0 < \delta \leq s_k \leq l,$$

$$\max(\|[s]_n\|_{w_2}^2; \|[g]_n\|_{w_1}^2) \leq R^2\}$$

$$[s]_n = (s_0, s_1, \dots, s_n) \in \mathbb{R}^{n+1}, [g]_n = (g_0, g_1, \dots, g_n) \in \mathbb{R}^{n+1}$$

$$\|[s]_n\|_{w_2}^2 = \sum_{k=0}^{n-1} \tau s_k^2 + \sum_{k=1}^n \tau s_{\bar{t},k}^2 + \sum_{k=1}^{n-1} \tau s_{\bar{t}\bar{t},k}^2, \|[g]_n\|_{w_1}^2 = \sum_{k=0}^{n-1} \tau g_k^2 + \sum_{k=1}^n \tau g_{\bar{t},k}^2.$$

$$s_{\bar{t},k} = \frac{s_k - s_{k-1}}{\tau}, s_{t,k} = \frac{s_{k+1} - s_k}{\tau}, s_{\bar{t}\bar{t},k} = \frac{s_{k+1} - 2s_k + s_{k-1}}{\tau^2}.$$

Definition 1

$[u([v]_n)]_n = (u(x; 0), u(x; 1), \dots, u(x; n))$ is called discrete state vector if

1. $u(x; 0) = \phi(x) \in W_2^1[0, s_0]$;
2. $u(x; k) \in W_2^1[0, s_k]$ satisfy the integral identity

$$\int_0^{s_k} \left(a_k(x) \frac{du(x; k)}{dx} \eta'(x) - b_k \frac{du(x; k)}{dx} \eta(x) - c_k(x) u(x; k) \eta(x) + f_k(x) \eta + u_{\bar{t}}(x; k) \eta(x) \right) dx + \left((\gamma_{s^n} (s^n)')^k - \chi_{s^n}^k \right) \eta(s_k) + g_k^n \eta(0) = 0, \\ \forall \eta \in W_2^1[0, s_k] \quad (6)$$

3. $u(x; k) \in W_2^1[0, s_k]$ iteratively continued to $[0, l]$ as

$$u(x; k) = u(2^n s_k - x; k), \quad 2^{n-1} s_k \leq x \leq 2^n s_k, \quad n = \overline{1, n_k},$$

$$n_k \leq N = 1 + \log_2 \left[\frac{l}{\delta} \right] \quad (7)$$

Discrete Optimal Control Problem

$$\mathcal{I}_n([v]_n) = \beta_0 \tau \sum_{k=1}^n \left(u(0; k) - \nu_k \right)^2 + \beta_1 \tau \sum_{k=1}^n \left(u(s_k; k) - \mu_k \right)^2 \quad (8)$$

$$V_R^n = \{ [v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} : 0 < \delta \leq s_k \leq l, \\ \max(\|[s]_n\|_{w_2}^2; \|[g]_n\|_{w_1}^2) \leq R^2 \}$$

$[u([v]_n)]_n = (u(x; 0), u(x; 1), \dots, u(x; n))$ be a discrete state vector. Formulated discrete optimal control problem will be called Problem \mathcal{I}_n .

$$u^\tau(x, t) = u(x; k), \quad \text{if } t_{k-1} < t \leq t_k, \quad 0 \leq x \leq l, \quad k = \overline{1, n},$$

$$\hat{u}^\tau(x, t) = u(x; k-1) + u_\tau(x; k)(t - t_{k-1}), \quad \text{if } t_{k-1} < t \leq t_k, \quad 0 \leq x \leq l, \quad k = \overline{1, n}$$

$$\hat{u}^\tau(x, t) = u(x; n), \quad \text{if } t \geq T, \quad 0 \leq x \leq l.$$

$$u^\tau \in V_2(D), \quad \hat{u}^\tau \in W_2^{1,1}(D)$$

$$\begin{aligned} D &= \{(x, t) : 0 < x < l, 0 < t \leq T\} \\ a, b, c &\in L_\infty(D), f \in L_2(D), \\ \phi &\in W_2^1[0, s_0], \gamma, \chi \in W_2^{1,1}(D), \mu, \nu \in L_2[0, T], \\ \frac{\partial a}{\partial x} &\in L_\infty(D), \int_0^T \operatorname{ess\,sup}_{0 \leq x \leq l} \left| \frac{\partial a}{\partial t} \right| dt < +\infty. \end{aligned} \quad (9)$$

Theorem 2

The Problem \mathcal{I} has a solution, i.e.

$$V_* = \{v \in V_R : \mathcal{J}(v) = \mathcal{J}_* \equiv \inf_{v \in V_R} \mathcal{J}(v)\} \neq \emptyset$$

Convergence Theorem

Theorem 3

Sequence of discrete optimal control problems \mathcal{I}_n approximates the optimal control problem \mathcal{I} with respect to functional, i.e.

$$\lim_{n \rightarrow +\infty} I_{n_*} = J_*, \quad (10)$$

where

$$I_{n_*} = \inf_{V_R^n} I_n([v]_n), \quad n = 1, 2, \dots$$

If $[v]_{n_\epsilon} \in V_R^n$ is chosen such that

$$I_{n_*} \leq I_n([v]_{n_\epsilon}) \leq I_{n_*} + \epsilon_n, \quad \epsilon_n \downarrow 0,$$

then the sequence $v_n = (s_n, g_n) = \mathcal{P}_n([v]_{n_\epsilon})$ converges to some element $v_ = (s_*, g_*) \in V_*$ weakly in $W_2^2[0, T] \times W_2^1[0, T]$, and strongly in $W_2^1[0, T] \times L_2[0, T]$. In particular s_n converges to s_* uniformly on $[0, T]$. Moreover, piecewise linear interpolation \hat{u}^τ of the discrete state vector $[u[v]_{n_\epsilon}]_n$ converges to the solution $u(x, t; v_*) \in W_2^{1,1}(\Omega_*)$ of the Neumann problem (2)-(5) weakly in $W_2^{1,1}(\Omega_*)$.*

First Energy Estimate and its Consequences

Theorem 4

For all sufficiently small τ discrete state vector $[u([v]_n)]_n$ satisfies:

$$\max_{0 \leq k \leq n} \int_0^l u^2(x; k) dx + \tau \sum_{k=1}^n \int_0^l \left| \frac{du(x; k)}{dx} \right|^2 dx \leq$$

$$C \left(\|\phi\|_{L_2(0, s_0)}^2 + \|g^n\|_{L_2(0, T)}^2 + \|f\|_{L_2(D)}^2 + \|\gamma(s^n(t), t)(s^n)'(t)\|_{L_2(0, T)}^2 \right. \\ \left. + \|\chi(s^n(t), t)\|_{L_2(0, T)}^2 + \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx \right), \quad (11)$$

$$\max_{0 \leq k \leq n} \int_0^l u^2(x; k) dx + \tau \sum_{k=0}^n \int_0^l \left| \frac{du(x; k)}{dx} \right|^2 dx + \tau^2 \sum_{k=1}^n \int_0^l u_t^2(x; k) dx \leq$$

$$C \left(\|\phi\|_{W_2^1(0, s_0)}^2 + \|g^n\|_{L_2(0, T)}^2 + \|f\|_{L_2(D)}^2 + \|\gamma(s^n(t), t)(s^n)'(t)\|_{L_2(0, T)}^2 \right. \\ \left. + \|\chi(s^n(t), t)\|_{L_2(0, T)}^2 + \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx \right), \quad (12)$$

Theorem 5

Let $[v]_n \in V_R^n$, $n = 1, 2, \dots$ be a sequence of discrete controls and the sequence $\{\mathcal{P}_n([v]_n)\}$ converges strongly in $W_2^1[0, T] \times L_2[0, T]$ to $v = (s, g)$. Then the sequence $\{u^\tau\}$ converges as $\tau \rightarrow 0$ weakly in $W_2^{1,0}(\Omega)$ to weak solution $u \in V_2^{1,0}(\Omega)$ of the problem (2)-(5), i.e. to the solution of the integral identity (??). Moreover, u satisfies the energy estimate

$$\|u\|_{V_2^{1,0}(D)}^2 \leq C \left(\|\phi\|_{L_2(0,s_0)}^2 + \|g\|_{L_2(0,T)}^2 + \|f\|_{L_2(D)}^2 + \|\gamma\|_{W_2^{1,0}(D)}^2 + \|\chi\|_{W_2^{1,0}(D)}^2 \right) \quad (13)$$

Corollary 6

For arbitrary $v = (s, g) \in V_R$ there exists a weak solution $u \in V_2^{1,0}(\Omega)$ of the problem (2)-(5) which satisfy the energy estimate (6):

$$\|u\|_{V_2^{1,0}(D)}^2 \leq C \left(\|\phi\|_{L_2(0,s_0)}^2 + \|g\|_{L_2(0,T)}^2 + \|f\|_{L_2(D)}^2 + \|\gamma\|_{W_2^{1,0}(D)}^2 + \|\chi\|_{W_2^{1,0}(D)}^2 \right)$$

Second Energy Estimate and its Consequences

Theorem 7

For all sufficiently small τ discrete state vector $[u([v]_n)]_n$ satisfies the following stability estimation:

$$\begin{aligned} \max_{1 \leq k \leq n} \int_0^l \left| \frac{du(x; k)}{dx} \right|^2 dx + \tau \sum_{k=1}^n \int_0^l u_t^2(x; k) dx \leq C \left[\|\phi\|_{W_2^1[0, l]}^2 + \right. \\ \left. \|g^n\|_{W_2^{\frac{1}{4}}[0, T]}^2 + \|\gamma(s^n(t), t)(s^n)'(t)\|_{W_2^{\frac{1}{4}}[0, T]}^2 + \|\chi(s^n(t), t)\|_{W_2^{\frac{1}{4}}[0, T]}^2 + \right. \\ \left. + \|f\|_{L_2(D)}^2 + \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx \right] \quad (14) \end{aligned}$$

$W_2^{\frac{1}{4}}[0, T]$ – Banach space of all elements of $L_2[0, T]$ with finite norm

$$\|u\|_{W_2^{\frac{1}{4}}[0, T]} = \left(\|u\|_{L_2[0, T]}^2 + \int_0^T dt \int_0^T \frac{|u(t) - u(\tau)|^2}{|t - \tau|^{\frac{3}{2}}} d\tau \right)^{\frac{1}{2}}$$

Theorem 8

Let $[v]_n \in V_R^n$, $n = 1, 2, \dots$ be a sequence of discrete controls and the sequence $\{\mathcal{P}_n([v]_n)\}$ converges weakly in $W_2^2[0, T] \times W_2^1[0, T]$ to $v = (s, g)$. Then the sequence $\{\hat{u}^\tau\}$ converges as $\tau \rightarrow 0$ weakly in $W_2^{1,1}(\Omega)$ to weak solution $u \in W_2^{1,1}(\Omega)$ of the problem (2)-(5).

Moreover, u satisfies the energy estimate

$$\|u\|_{W_2^{1,1}(D)}^2 \leq C \left(\|\phi\|_{W_2^1(0, s_0)}^2 + \|g\|_{W_2^{\frac{1}{4}}}^2 + \|f\|_{L_2(D)}^2 + \|\gamma\|_{W_2^{1,1}(D)}^2 + \|\chi\|_{W_2^{1,1}(D)}^2 \right) \quad (15)$$

Corollary 9

For arbitrary $v = (s, g) \in V_R$ there exists a weak solution $u \in W_2^{1,1}(\Omega)$ of the problem (2)-(5) which satisfy the energy estimate

$$\|u\|_{W_2^{1,1}(D)}^2 \leq C \left(\|\phi\|_{W_2^1(0, s_0)}^2 + \|g\|_{W_2^{\frac{1}{4}}}^2 + \|f\|_{L_2(D)}^2 + \|\gamma\|_{W_2^{1,1}(D)}^2 + \|\chi\|_{W_2^{1,1}(D)}^2 \right)$$

Proof of Convergence Theorem

Lemma 10

Let $\mathcal{J}_*(\pm\epsilon) = \inf_{V_{R\pm\epsilon}} \mathcal{J}(v)$, $\epsilon > 0$. Then

$$\lim_{\epsilon \rightarrow 0} \mathcal{J}_*(\epsilon) = \mathcal{J}_* = \lim_{\epsilon \rightarrow 0} \mathcal{J}_*(-\epsilon) \quad (16)$$

Lemma 11

For arbitrary $v = (s, g) \in V_R$,

$$\lim_{n \rightarrow \infty} \mathcal{I}_n(\mathcal{Q}_n(v)) = \mathcal{J}(v) \quad (17)$$

Lemma 12

For arbitrary $[v]_n \in V_R^n$

$$\lim_{n \rightarrow \infty} \left(\mathcal{J}(\mathcal{P}_n([v]_n)) - \mathcal{I}_n([v]_n) \right) = 0 \quad (18)$$

Proof of Convergence Theorem

Proof of Lemma 12

Let $[v]_n \in V_R^n$ and $v^n = (s^n, g^n) = \mathcal{P}_n([v]_n)$. From Lemma 2.3 it follows that the sequence $\{\mathcal{P}_n([v]_n)\}$ is weakly precompact in $W_2^2[0, T] \times W_2^1[0, T]$. Assume that the whole sequence converges to $\tilde{v} = (\tilde{s}, \tilde{g})$ weakly in $W_2^2[0, T] \times W_2^1[0, T]$. This implies the strong convergence in $W_2^1[0, T] \times L_2[0, T]$; $\tilde{v} \in V_R$. Let $\mathcal{Q}_n(\tilde{v}) = [\tilde{v}]_n$.

$$\mathcal{I}_n([v]_n) - \mathcal{J}(v^n) = \mathcal{I}_n([v]_n) - \mathcal{I}_n([\tilde{v}]_n) + \mathcal{I}_n([\tilde{v}]_n) - \mathcal{J}(\tilde{v}) + \mathcal{J}(\tilde{v}) - \mathcal{J}(v^n) \quad (19)$$

We proved the weak continuity of the functional $\mathcal{J}(v)$, i.e.

$$\lim_{n \rightarrow \infty} (\mathcal{J}(\tilde{v}) - \mathcal{J}(v^n)) = 0.$$

From Lemma 11 it follows that

$$\lim_{n \rightarrow \infty} (\mathcal{I}_n([\tilde{v}]_n) - \mathcal{J}(\tilde{v})) = 0.$$

Hence, we only need to prove that

$$\lim_{n \rightarrow \infty} (\mathcal{I}_n([v]_n) - \mathcal{I}_n([\tilde{v}]_n)) = 0 \quad (20)$$

Proof of Convergence Theorem

Proof of Lemma 12, cont'd

$$[u([v]_n)]_n = (u_n(x; 0), u_n(x; 1), \dots, u_n(x; n)),$$

$$[u([\tilde{v}]_n)]_n = (\tilde{u}(x; 0), \tilde{u}(x; 1), \dots, \tilde{u}(x; n))$$

$$s_k = s^n(t_k), \tilde{s}_k = \tilde{s}(t_k), \Delta u(x; k) = u_n(x; k) - \tilde{u}(x; k)$$

$$\begin{aligned} \mathcal{I}_n([v]_n) - \mathcal{I}_n([\tilde{v}]_n) &= \beta_0 \sum_{k=1}^n \tau (u_n(0; k) - f_0^k)^2 + \beta_1 \sum_{k=1}^n \tau (u_n(s_k; k) - f_1^k)^2 \\ &- \beta_0 \sum_{k=1}^n \tau (\tilde{u}(0; k) - f_0^k)^2 - \beta_1 \sum_{k=1}^n \tau (\tilde{u}(\tilde{s}_k; k) - f_1^k)^2 = \beta_0 \sum_{k=1}^n \tau (\Delta u(0; k))^2 + \\ &+ 2\beta_0 \sum_{k=1}^n \tau \Delta u(0; k) (\tilde{u}(0; k) - f_0^k) + \beta_1 \sum_{k=1}^n \tau (\Delta u(s_k; k) + \tilde{u}(s_k; k) - \tilde{u}(\tilde{s}_k; k))^2 \\ &+ 2\beta_1 \sum_{k=1}^n \tau (\Delta u(s_k; k) + \tilde{u}(s_k; k) - \tilde{u}(\tilde{s}_k; k)) (\tilde{u}(\tilde{s}_k; k) - f_1^k) \quad (21) \end{aligned}$$

Proof of Convergence Theorem

Proof of Lemma 12, cont'd

$$\begin{aligned} \sum_{k=1}^n \tau (\tilde{u}(s_k; k) - \tilde{u}(\tilde{s}_k; k))^2 &= \sum_{k=1}^n \tau \left| \int_{s_k}^{\tilde{s}_k} \frac{d\tilde{u}(x; k)}{dx} dx \right|^2 \\ &\leq \max_{1 \leq k \leq n} |s_k - \tilde{s}_k| \sum_{k=1}^n \tau \left| \int_{s_k}^{\tilde{s}_k} \left| \frac{\partial \tilde{u}(x; k)}{\partial x} \right|^2 dx \right| \rightarrow 0, \quad \text{as } n \rightarrow +\infty. \end{aligned} \quad (22)$$

It is enough to prove that

$$R = \sum_{k=1}^n \tau \left[(\Delta u(0; k))^2 + (\Delta u(s_k; k))^2 \right] \rightarrow 0 \text{ as } n \rightarrow +\infty \quad (23)$$

By the Morrey inequality

$$R \leq C \sum_{k=1}^n \tau \left[\int_0^{s_k} |\Delta u(x; k)|^2 dx + \int_0^{s_k} \left| \frac{d\Delta u(x; k)}{dx} \right|^2 dx \right] \quad (24)$$

where C is independent of n .

Proof of Convergence Theorem

Proof of Lemma 12, cont'd

Let us subtract integral identities for $u_n(x; k)$ and $\tilde{u}(x; k)$, by assuming that the fixed test function η belongs to $W_2^1[0, l]$.

$$\begin{aligned} & \int_0^{s_k} \left(a_k(x) \frac{d\Delta u}{dx} \frac{d\eta}{dx} - b_k(x) \frac{d\Delta u}{dx} \eta(x) - c_k(x) \Delta u \eta + \Delta u_{\bar{t}} \eta \right) dx - \\ & \chi_{\tilde{s}}^k [\eta(s_k) - \eta(\tilde{s}_k)] + \int_{s_k}^{\tilde{s}_k} \left(a_k(x) \frac{d\tilde{u}}{dx} \frac{d\eta}{dx} - b_k(x) \frac{d\tilde{u}}{dx} \eta - c_k(x) \tilde{u} \eta + \right. \\ & \left. f_k(x) \eta + \tilde{u}_{\bar{t}} \eta \right) dx + (g_k^n - \tilde{g}_k^n) \eta(0) + [(\gamma_{s^n} (s^n)')^k - (\gamma_{\tilde{s}} \tilde{s}')^k] \eta(s_k) + \\ & (\gamma_{\tilde{s}} \tilde{s}')^k [\eta(s_k) - \eta(\tilde{s}_k)] - [\chi_{s^n}^k - \chi_{\tilde{s}}^k] \eta(s_k) = 0 \end{aligned} \quad (25)$$

Proof of Convergence Theorem

Proof of Lemma 12, cont'd

Our goal now is to derive from (25) that the right-hand side of (24) converges to zero as $n \rightarrow +\infty$. The proof goes along the same lines as the derivation of the first energy estimate:

$$\begin{aligned} & a_0 \tau \int_0^{s_k} \left| \frac{d\Delta u(x; k)}{dx} \right|^2 dx + \int_0^{s_k} \Delta u^2(x; k) dx - \int_0^{s_k} \Delta u^2(s; k-1) dx + \\ & \quad \tau^2 \int_0^{s_k} \Delta u_{\bar{t}}^2(x; k) dx \leq C_1 \tau \left[\int_0^{s_k} \Delta u^2(x; k) dx + |g_k^n - \tilde{g}_k^n|^2 \right] - \\ & 2\tau \int_{s_k}^{\tilde{s}_k} \left[a_k(x) \frac{d\tilde{u}}{dx} \frac{d\Delta u(x; k)}{dx} - b_k(x) \frac{d\tilde{u}}{dx} \Delta u(x; k) - c_k(x) \tilde{u} \Delta u(x; k) + f_k(x) \times \right. \\ & \quad \left. \times \Delta u(x; k) + \tilde{u}_{\bar{t}} \Delta u(x; k) \right] dx - 2 \int_{t_{k-1}}^{t_k} \left(\gamma(s^n(t), t) (s^n)'(t) - \gamma(\tilde{s}(t), t) \tilde{s}'(t) \right) dt \times \\ & \quad \times \Delta u(s_k; k) - 2 \int_{t_{k-1}}^{t_k} \int_{\tilde{s}_k}^{s_k} \gamma(\tilde{s}(t), t) \tilde{s}'(t) \frac{d\Delta u(x; k)}{dx} dx dt + 2 \int_{t_{k-1}}^{t_k} \left(\chi(s^n(t), t) - \right. \\ & \quad \left. \chi(\tilde{s}(t), t) \right) dt \Delta u(s_k; k) + 2 \int_{t_{k-1}}^{t_k} \int_{\tilde{s}_k}^{s_k} \chi(\tilde{s}(t), t) \frac{d\Delta u(x; k)}{dx} dx dt \quad (26) \end{aligned}$$

Proof of Convergence Theorem

Proof of Lemma 12, cont'd

By applying the technique developed along with the proof of the first energy est. from (26) it follows that for all sufficiently small τ

$$\begin{aligned} \max_{1 \leq k \leq n} \int_0^{s_k} \Delta u^2(x; k) dx &\leq C \left(\|g^n - \tilde{g}\|_{L_2[0, T]}^2 + \right. \\ &\left. \sum_{j=1}^{n-1} \mathbf{1}_{+(s_{j+1} - s_j)} \int_{s_j}^{s_{j+1}} \Delta u^2(x; j) dx + \sum_{j=1}^n |\mathcal{L}_j| \right) \end{aligned} \quad (27)$$

$$\begin{aligned} \mathcal{L}_j = \tau \int_{s_j}^{\tilde{s}_j} &\left(a_j(x) \frac{d\tilde{u}}{dx} \frac{d\Delta u(x; j)}{dx} - b_j(x) \frac{d\tilde{u}}{dx} \Delta u(x; j) - c_j(x) \tilde{u} \Delta u(x; j) + \right. \\ &\left. f_j(x) \Delta u(x; j) + \tilde{u}_{\tilde{t}} \Delta u(x; j) \right) dx + \int_{t_{j-1}}^{t_j} [\gamma(s^n(t), t)(s^n)'(t) - \gamma(\tilde{s}(t), t)\tilde{s}'(t)] dt \\ &\times \Delta u(s_j; j) + \int_{t_{j-1}}^{t_j} \int_{\tilde{s}_j}^{s_j} \gamma(\tilde{s}(t), t)\tilde{s}'(t) \frac{d\Delta u(x; j)}{dx} dx dt + \int_{t_{j-1}}^{t_j} \left(\chi(s^n(t), t) - \right. \\ &\left. \chi(\tilde{s}(t), t) \right) dt \Delta u(s_j; j) + \int_{t_{j-1}}^{t_j} \int_{\tilde{s}_j}^{s_j} \chi(\tilde{s}(t), t) \frac{d\Delta u(x; j)}{dx} dx dt \end{aligned} \quad (28)$$

Proof of Convergence Theorem

Proof of Lemma 12, cont'd

Having (27) we perform summation in (26) with respect to k from 1 to n and derive

$$\begin{aligned} & \max_{1 \leq k \leq n} \int_0^{s_k} \Delta u^2(x; k) dx + \sum_{k=1}^n \tau \int_0^{s_k} \left| \frac{d\Delta u(x; k)}{dx} \right|^2 dx + \\ & \sum_{k=1}^n \tau^2 \int_0^{s_k} \Delta u_t^2(x; k) dx \leq C_1 \left(\|g^n - \tilde{g}\|_{L_2[0, T]}^2 + \right. \\ & \left. \sum_{j=1}^{n-1} \mathbf{1}_+(s_{j+1} - s_j) \int_{s_j}^{s_{j+1}} \Delta u^2(x; j) dx + \sum_{j=1}^n |\mathcal{L}_j| \right) \quad (29) \end{aligned}$$

$$\Omega_n^1 = \{0 < t < T, 0 < x < \tilde{s}_1^n(t)\}, \quad \Omega_n = \bigcup_{k=1}^n \{t_{k-1} < t \leq t_k, 0 < x < s_k\}$$

Proof of Convergence Theorem

Proof of Lemma 12, cont'd

$$\begin{aligned}\|\Delta u^\tau\|_{V_2^{1,0}(\Omega_n)}^2 &= \max_{1 \leq k \leq n} \int_0^{s_k} \Delta u^2(x; k) dx + \sum_{k=1}^n \tau \int_0^{s_k} \left| \frac{d\Delta u(x; k)}{dx} \right|^2 dx \\ \|\Delta u^\tau\|_{V_2^{1,0}(\Omega_n)}^2 &\leq C_2 \left(\|g^n - \tilde{g}\|_{L_2[0,T]}^2 + \|\Delta u^\tau\|_{V_2^{1,0}(\Omega_n^1 - \Omega_n)}^2 + \sum_{j=1}^n |\mathcal{L}_j| \right)\end{aligned}\tag{30}$$

Since

$$\|g^n - \tilde{g}\|_{L_2[0,T]} \rightarrow 0, \text{ as } n \rightarrow +\infty,$$

it is enough to prove that

$$\lim_{n \rightarrow +\infty} \sum_{j=1}^n |\mathcal{L}_j| = 0.\tag{31}$$

Proof of Convergence Theorem

Proof of Lemma 12, cont'd

$$\begin{aligned} & \sum_{j=1}^n \left| \tau \int_{s_j}^{\tilde{s}_j} a_j(x) \frac{d\tilde{u}(x; j)}{dx} \frac{d\Delta u(x; j)}{dx} dx \right| = \\ & \sum_{j=1}^n \left| \int_{t_{j-1}}^{t_j} \int_{s_j}^{\tilde{s}_j} a(x, t) \frac{d\tilde{u}^\tau}{dx} \frac{d\Delta u^\tau}{dx} dx \right| \leq C \left\| \frac{\partial \tilde{u}^\tau}{\partial x} \right\|_{L_2(\tilde{\Delta})} \left\| \frac{\partial \Delta u^\tau}{\partial x} \right\|_{L_2(\tilde{\Delta})} \end{aligned} \quad (32)$$

$$\tilde{\Delta} = \bigcup_{j=1}^n \{(x, t) : t_{j-1} < t < t_j, \min(s_j, \tilde{s}_j) < x < \max(s_j, \tilde{s}_j)\}$$

$$\sum_{j=1}^n \left| \tau \int_{s_j}^{\tilde{s}_j} \tilde{u}_{\bar{t}}(x; j) \Delta u(x; j) dx \right| \leq \left(\sum_{j=1}^n \tau \int_0^l \tilde{u}_{\bar{t}}^2(x; j) dx \right)^{\frac{1}{2}} \|\Delta u^\tau\|_{L_2(\tilde{\Delta})} \quad (33)$$

$$\begin{aligned}
& \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left| \left(\gamma(s^n(t), t)(s^n)'(t) - \gamma(\tilde{s}(t), t)\tilde{s}'(t) \right) \Delta u(s_j, j) \right| dt \leq \\
& \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left| \left(\gamma(s^n(t), t)(s^n)'(t) - \gamma(\tilde{s}(t), t)\tilde{s}'(t) \right) \left(\int_{\tilde{s}(t)}^{s_j} \frac{\partial \Delta u^\tau(x, t)}{\partial x} dx + \right. \right. \\
& \left. \left. \Delta u(\tilde{s}(t); j) \right) \right| dt \leq \|\gamma(s^n(t), t) - \gamma(\tilde{s}(t), t)\|_{L_2[0, T]} \|(s^n)'\|_{C[0, T]} \|\Delta u^\tau(\tilde{s}(t), t)\|_{L_2} \\
& \quad + \|(s^n)'(t) - \tilde{s}'(t)\|_{L_2[0, T]} \|\gamma(\tilde{s}(t), t)\|_{L_4[0, T]} \|\Delta u^\tau(\tilde{s}(t), t)\|_{L_4[0, T]} + \\
& \sum_{j=1}^n \left(\int_{t_{j-1}}^{t_j} |\gamma(s^n(t), t)(s^n)'(t) - \gamma(\tilde{s}(t), t)\tilde{s}'(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_{t_{j-1}}^{t_j} \left| \int_{\tilde{s}(t)}^{s_j} \frac{\partial \Delta u^\tau}{\partial x} dx \right|^2 \right)^{\frac{1}{2}} \\
& \leq C \|\gamma(s^n(t), t) - \gamma(\tilde{s}(t), t)\|_{L_2[0, T]} \|(s^n)'\|_{W_2^1[0, T]} \|\Delta u^\tau\|_{W_2^{1,0}(D)} + \\
& C \|(s^n)'(t) - \tilde{s}'(t)\|_{L_2[0, T]} \|\gamma\|_{W_2^{1,1}(D)} \|\Delta u^\tau\|_{V_2(D)} + \left\| \frac{\partial \Delta u^\tau}{\partial x} \right\|_{L_2(\tilde{\Delta})} \times \\
& \|\gamma(s^n(t), t)(s^n)'(t) - \gamma(\tilde{s}(t), t)\tilde{s}'(t)\|_{L^2[0, T]} \left(\max_{0 \leq t \leq T} |\tilde{s}(t) - s^n(t)| \right)^{\frac{1}{2}}
\end{aligned} \tag{34}$$

► U. G. Abdulla.

On the Optimal Control of the Free Boundary Problems for the Second Order Parabolic Equations. I. Well-posedness and Convergence of the Method of Lines.

Inverse Problems and Imaging, 7(2):307–340, May 2013.

Method of Finite Differences

Spatial grid

Given $[v]_n \in V_R^n$, let (p_0, p_1, \dots, p_n) be a permutation of $(0, 1, \dots, n)$ according to order

$$s_{p_0} \leq s_{p_1} \leq \dots \leq s_{p_n}$$

$$\forall k \in \{0, 1, \dots, n\} \exists! j_k \text{ such that } s_k = s_{p_{j_k}}; \quad j := j_k \quad (35)$$

$\omega_{p_0} = \{x_i = i \cdot h, i = 0, 1, \dots, m_0^{(n)}\}$ be a grid on $[0, s_{p_0}]$; $h = \frac{s_{p_0}}{m_0^{(n)}}$

$\omega_{p_k} = \{x_i : i = 0, 1, \dots, m_k^{(n)}\}$ be a grid on $[0, s_{p_k}]$; $m_k^{(n)} \geq m_{k-1}^{(n)}$

if $s_{p_n} < l$, then introduce a grid on $[s_{p_n}, l]$: $\bar{\omega} = \{x_i : i = m_n^{(n)}, \dots, N\}$

$$h_i = x_{i+1} - x_i, \quad i = 0, 1, \dots, N-1;$$

$$m_k^{(n)} \equiv m_k, \quad m_k \rightarrow +\infty, \quad \max_{0 \leq i \leq N-1} h_i = O(\sqrt{\tau}), \quad \text{as } n \rightarrow \infty. \quad (36)$$

Method of Finite Differences

Steklov averages

$$d_k(x) = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} d(x, t) dt, \quad h_k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} h(t) dt,$$

$$d_{ik} = \frac{1}{h_i \tau} \int_{x_i}^{x_{i+1}} \int_{t_{k-1}}^{t_k} d(x, t) dt dx,$$

where $i = 0, 1, \dots, N - 1$; $k = 1, \dots, n$; d stands for any of the functions a, b, c, f , and h stands for any of the functions ν, μ, g or g^n .

Given $v = (s, g) \in V_R$ we define Steklov averages of traces

$$\chi_s^k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \chi(s(t), t) dt, \quad (\gamma_s s')^k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \gamma(s(t), t) s'(t) dt. \quad (37)$$

Given $[v]_n = ([s]_n, [g]_n) \in V_R^n$ we define Steklov averages $\chi_{s^n}^k$ and $(\gamma_{s^n} (s^n)')^k$ through (37) with s replaced by s^n .

$$[u([v]_n)]_n = (u(0), u(1), \dots, u(n)), \quad u(k) \in \mathbb{R}^{N+1}, \quad k = 0, \dots, n$$

is called a discrete state vector if

(1) $u_i(0) = \frac{1}{h_i} \int_{x_i}^{x_{i+1}} \phi(x) dx, \quad i = 0, 1, \dots, m_0;$

(2) $u_i(k), k = 1, \dots, n; i = 0, \dots, m_j$ satisfy system of LAEs:

$$\begin{aligned} & \left[a_{0k} + hb_{0k} - h^2 c_{0k} + \frac{h^2}{\tau} \right] u_0(k) - \left[a_{0k} + hb_{0k} \right] u_1(k) = \\ & \quad \frac{h^2}{\tau} u_0(k-1) - h^2 f_{0k} - hg_k^n, \\ & -a_{i-1,k} h_i u_{i-1}(k) + \left[a_{i-1,k} h_i + a_{ik} h_{i-1} + b_{ik} h_i h_{i-1} - c_{ik} h_i^2 h_{i-1} + \right. \\ & \quad \left. \frac{h_i^2 h_{i-1}}{\tau} \right] u_i(k) - \left[a_{ik} h_{i-1} + b_{ik} h_i h_{i-1} \right] u_{i+1}(k) = \\ & \quad -h_i^2 h_{i-1} f_{ik} + \frac{h_i^2 h_{i-1}}{\tau} u_i(k-1), \quad i = 1, \dots, m_j - 1 \\ & -a_{m_j-1,k} u_{m_j-1}(k) + a_{m_j-1,k} u_{m_j}(k) = -h_{m_j-1} \left[(\gamma_{s^n} (s^n)')^k - \chi_{s^n}^k \right]. \end{aligned} \tag{38}$$

(3) For arbitrary $k = 0, 1, \dots, n$, the remaining components of $u(k) \in \mathbb{R}^{N+1}$ are calculated as

$$u_i(k) = \hat{u}(x_i; k), \quad m_j \leq i \leq N$$

where $\hat{u}(x; k) \in W_2^1[0, l]$ is a piecewise linear interpolation of $\{u_i(k) : i = 0, \dots, m_j\}$, that is to say

$$\hat{u}(x; k) = u_i(k) + \frac{u_{i+1}(k) - u_i(k)}{h_i} (x - x_i), \quad x_i \leq x \leq x_{i+1}, \quad i = 0, \dots, m_j - 1,$$

iteratively continued to $[0, l]$ as

$$\begin{aligned} \hat{u}(x; k) &= \hat{u}(2^n s_k - x; k), \quad 2^{n-1} s_k \leq x \leq 2^n s_k, \quad n = \overline{1, n_k}, \\ n_k &\leq n_* = 1 + \log_2 \left[\frac{l}{\delta} \right] \end{aligned} \quad (39)$$

where $[r]$ means integer part of the real number r .

Discrete State Vector

derivation

$$\int_0^{s_k} \left(a_k(x) \frac{du(x; k)}{dx} \eta'(x) - b_k \frac{du(x; k)}{dx} \eta(x) - c_k(x) u(x; k) \eta(x) + f_k(x) \eta + u_{\bar{t}}(x; k) \eta(x) \right) dx + \left((\gamma_{s^n} (s^n)')^k - \chi_{s^n}^k \right) \eta(s_k) + g_k^n \eta(0) = 0, \\ \forall \eta \in W_2^1[0, s_k] \quad (40)$$

$$\sum_{i=0}^{m_j-1} h_i \left(a_{ik} u_{ix}(k) \eta_{ix}(k) - b_{ik} u_{ix}(k) \eta_i(k) - c_{ik} u_i(k) \eta_i(k) + f_{ik} \eta_i(k) + u_{i\bar{t}}(k) \eta_i(k) \right) + \left((\gamma_{s^n} (s^n)')^k - \chi_{s^n}^k \right) \eta_{m_j}(k) + g_k^n \eta_0(k) = 0, \\ \forall \eta_i(k), i = 0, \dots, m_j \quad (41)$$

Discrete State Vector

derivation

By using summation by parts, we have

$$\begin{aligned} \sum_{i=1}^{m_j-1} h_i a_{ik} u_{ix}(k) \eta_{ix}(k) &= \sum_{i=0}^{m_j-1} a_{ik} u_{ix}(k) \eta_{i+1}(k) - \sum_{i=0}^{m_j-1} a_{ik} u_{ix}(k) \eta_i(k) \\ &= \sum_{i=1}^{m_j} a_{i-1,k} u_{i-1,x}(k) \eta_i(k) - \sum_{i=0}^{m_j-1} a_{ik} u_{ix}(k) \eta_i(k) = \\ &\quad \sum_{i=1}^{m_j-1} \left(a_{i-1,k} u_{i-1,x}(k) - a_{ik} u_{ix}(k) \right) \eta_i(k) + \\ &\quad a_{m_j-1,k} u_{m_j-1,x}(k) \eta_{m_j}(k) - a_{0k} u_{0x}(k) \eta_0(k). \end{aligned} \quad (42)$$

Taking into account in (41):

Discrete State Vector

derivation

$$\begin{aligned} & \sum_{i=1}^{m_j-1} h_i \left(\frac{1}{h_i} \left(a_{i-1,k} u_{i-1,x}(k) - a_{ik} u_{ix}(k) \right) \right. \\ & \left. - b_{ik} u_{ix}(k) - c_{ik} u_i(k) + f_{ik} + u_{i\bar{t}}(k) \right) \eta_i(k) + \\ & \left(a_{m_j-1,k} u_{m_j-1,x}(k) + \left((\gamma_{s^n} (s^n)')^k - \chi_{s^n}^k \right) \right) \eta_{m_j}(k) + \\ & \left(-a_{0k} u_{0,x}(k) + h(-b_{0k} u_{0x}(k) - c_{0k} u_0(k) + f_{0k} + u_{0\bar{t}}(k)) \right) \eta_0(k) \\ & + g_k^n \eta_0(k) = 0 \end{aligned} \quad (43)$$

Since $\eta_i(k)$'s are arbitrary, system of LAE from the definition of the discrete state vector follows.

Fully Discretized Optimal Control Problem

$$\mathcal{I}_n([v]_n) = \beta_0 \tau \sum_{k=1}^n \left(u_0(k) - \nu_k \right)^2 + \beta_1 \tau \sum_{k=1}^n \left(u_{m_{j_k}}(k) - \mu_k \right)^2 \rightarrow \min$$

on a control set

$$V_R^n = \{[v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} : 0 < \delta \leq s_k \leq l, \\ \max(\|[s]_n\|_{w_2^2}; \|[g]_n\|_{w_1^2}) \leq R^2\}$$

Formulated discrete optimal control problem will be called Problem I_n .

$$u^\tau(x, t) = \hat{u}(x; k), \quad \text{if } t_{k-1} < t \leq t_k, \quad 0 \leq x \leq l, \quad k = \overline{0, n},$$

$$\hat{u}^\tau(x, t) = \hat{u}(x; k-1) + \hat{u}_{\bar{t}}(x; k)(t - t_{k-1}), \quad \text{if } t_{k-1} < t \leq t_k, \quad 0 \leq x \leq l, \quad k = \overline{1, n},$$

$$\hat{u}^\tau(x, t) = \hat{u}(x; n), \quad \text{if } t \geq T, \quad 0 \leq x \leq l.$$

$$\tilde{u}^\tau(x, t) = u_i(k), \quad \text{if } t_{k-1} < t \leq t_k, \quad x_i \leq x < x_{i+1}, \quad k = \overline{1, n}, \quad i = \overline{0, N-1}.$$

$$u^\tau \in V_2^{1,0}(D), \quad \hat{u}^\tau \in W_2^{1,1}(D), \quad \tilde{u}^\tau \in L_2(D).$$

$$u_{ix}(k) = \frac{u_{i+1}(k) - u_i(k)}{h_i}, \quad u_{i\bar{t}} = \frac{u_i(k) - u_i(k-1)}{\tau}, \quad \text{etc.}$$

$$D = \{(x, t) : 0 < x < l, 0 < t \leq T\}$$

$$a, b, c \in L_\infty(D), f \in L_2(D),$$

$$\phi \in W_2^1[0, s_0], \gamma, \chi \in W_2^{1,1}(D), \mu, \nu \in L_2[0, T],$$

$$\frac{\partial a}{\partial x} \in L_\infty(D), \int_0^T \operatorname{ess\,sup}_{0 \leq x \leq l} \left| \frac{\partial a}{\partial t} \right| dt < +\infty. \quad (44)$$

$$V_* = \{v \in V_R : \mathcal{J}(v) = \mathcal{J}_* \equiv \inf_{v \in V_R} \mathcal{J}(v)\}$$

Theorem 13

Sequence of discrete optimal control problems I_n approximates the optimal control problem I with respect to functional, i.e.

$$\lim_{n \rightarrow +\infty} \mathcal{I}_{n_*} = \mathcal{J}_*, \quad (45)$$

where

$$\mathcal{I}_{n_*} = \inf_{V_R^n} \mathcal{I}_n([v]_n), \quad n = 1, 2, \dots$$

If $[v]_{n_\epsilon} \in V_R^{n_\epsilon}$ is chosen such that






$$\mathcal{I}_{n_*} \leq \mathcal{I}_{n_\epsilon}([v]_{n_\epsilon}) \leq \mathcal{I}_{n_*} + \epsilon_n, \quad \epsilon_n \downarrow 0,$$









then the sequence $v_n = (s_n, g_n) = \mathcal{P}_n([v]_{n_\epsilon})$ converges to some element $v_* = (s_*, g_*) \in V_*$ weakly in $W_2^2[0, T] \times W_2^1[0, T]$, and strongly in $W_2^1[0, T] \times L_2[0, T]$. In particular s_n converges to s_* uniformly on $[0, T]$. Moreover, piecewise linear interpolation \hat{u}^τ of the discrete state vector $[u[v]_{n_\epsilon}]_n$ converges to the solution $u(x, t; v_*) \in W_2^{1,1}(\Omega_*)$ of the Neumann problem (2)-(5) weakly in $W_2^{1,1}(\Omega_*)$.







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On the Optimal Control of the Free Boundary Problems for the Second Order Parabolic Equations. II. Convergence of the Method of Finite Differences.







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