# Optimal Control and Inverse Problems for PDEs. Inverse Stefan Problem - Part 3

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# Inverse Stefan Problem

**Inverse Stefan Problem (ISP):** *Find the functions $u(x,t)$ and $s(t)$ and the boundary heat flux $g(t)$ satisfying conditions*

$$(a(x,t)u_x)_x + b(x,t)u_x + c(x,t)u - u_t = f(x,t), \quad \text{for } (x,t) \in \Omega$$

$$u(x,0) = \phi(x), \qquad 0 \le x \le s(0) = s_0$$

$$a(0,t)u_x(0,t) = g(t), \qquad 0 \le t \le T$$

$$a(s(t),t)u_x(s(t),t) + \gamma(s(t),t)s'(t) = \chi(s(t),t), \qquad 0 \le t \le T$$

$$u(s(t),t) = \mu(t), \qquad 0 \le t \le T$$

$$u(0,t) = \nu(t), \quad \text{for } 0 \le t \le T$$

# Optimal Control Problem

$$\mathcal{J}(v) = \beta_0 \|u(0,t) - \nu(t)\|^2_{L_2[0,T]} + \beta_1 \|u(s(t),t) - \mu(t)\|^2_{L_2[0,T]} \quad (1)$$

$$V_R = \Big\{ v = (s,g) \in W_2^2[0,T] \times W_2^1[0,T] : \delta \le s(t) \le l, \\ s(0) = s_0, \max( \|s\|_{W_2^2}; \ \|g\|_{W_2^1}) \le R \Big\}$$

$$(a(x,t)u_x)_x + b(x,t)u_x + c(x,t)u - u_t = f(x,t), \quad \text{for } (x,t) \in \Omega \quad (2)$$

$$u(x,0) = \phi(x), \qquad 0 \le x \le s(0) = s_0 \quad (3)$$

$$a(0,t)u_x(0,t) = g(t), \qquad 0 \le t \le T \quad (4)$$

$$a(s(t),t)u_x(s(t),t) + \gamma(s(t),t)s'(t) = \chi(s(t),t), \qquad 0 \le t \le T \quad (5)$$

$$\Omega = \{(x,t) : \ 0 < x < s(t), \ 0 < t \le T\}$$

# Discrete Optimal Control Problem

$$\omega_\tau = \{t_j = j \cdot \tau,\ j = 0, 1, \ldots, n\}$$

$$V_R^n = \{[v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} :\ 0 < \delta \leq s_k \leq l,$$

$$\max(\|[s]_n\|^2_{w_2^2};\ \|[g]_n\|^2_{w_2^1}) \leq R^2\}$$

$$[s]_n = (s_0, s_1, ..., s_n) \in \mathbb{R}^{n+1},\ [g]_n = (g_0, g_1, ..., g_n) \in \mathbb{R}^{n+1}$$

$$\|[s]_n\|^2_{w_2^2} = \sum_{k=0}^{n-1} \tau s_k^2 + \sum_{k=1}^{n} \tau s_{\bar{t},k}^2 + \sum_{k=1}^{n-1} \tau s_{\bar{t}t,k}^2,\ \|[g]_n\|^2_{w_2^1} = \sum_{k=0}^{n-1} \tau g_k^2 + \sum_{k=1}^{n} \tau g_{\bar{t},k}^2.$$

$$s_{\bar{t},k} = \frac{s_k - s_{k-1}}{\tau},\ s_{t,k} = \frac{s_{k+1} - s_k}{\tau},\ s_{\bar{t}t,k} = \frac{s_{k+1} - 2s_k + s_{k-1}}{\tau^2}.$$

# Discrete State Vector

## Definition 1

$[u([v]_n)]_n = (u(x;0), u(x;1), ..., u(x;n))$ is called discrete state vector if

1. $u(x;0) = \phi(x) \in W_2^1[0, s_0]$;
2. $u(x;k) \in W_2^1[0, s_k]$ satisfy the integral identity

$$\int_0^{s_k} \Big( a_k(x) \frac{du(x;k)}{dx} \eta'(x) - b_k \frac{du(x;k)}{dx} \eta(x) - c_k(x) u(x;k) \eta(x) + f_k(x) \eta + u_{\overline{t}}(x;k)\eta(x) \Big) dx + \Big( (\gamma_{s^n}(s^n)')^k - \chi_{s^n}^k \Big) \eta(s_k) + g_k^n \eta(0) = 0, \quad \forall \eta \in W_2^1[0, s_k] \tag{6}$$

3. $u(x;k) \in W_2^1[0, s_k]$ iteratively continued to $[0, l]$ as

$$u(x;k) = u(2^n s_k - x; k), \;\; 2^{n-1} s_k \leq x \leq 2^n s_k, n = \overline{1, n_k},$$

$$n_k \leq N = 1 + \log_2 \left[ \frac{l}{\delta} \right] \tag{7}$$

# Discrete Optimal Control Problem

$$\mathcal{I}_n([v]_n) = \beta_0 \tau \sum_{k=1}^{n} \Big( u(0;k) - \nu_k \Big)^2 + \beta_1 \tau \sum_{k=1}^{n} \Big( u(s_k;k) - \mu_k \Big)^2 \qquad (8)$$

$$V_R^n = \{[v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} : \ 0 < \delta \le s_k \le l,$$
$$\max(\|[s]_n\|^2_{w_2^2}; \ \|[g]_n\|^2_{w_2^1}) \le R^2\}$$

$[u([v]_n)]_n = (u(x;0), u(x;1), ..., u(x;n))$ be a discrete state vector.
Formulated discrete optimal control problem will be called Problem $\mathcal{I}_n$.

$$u^\tau(x,t) = u(x;k), \quad \text{if } t_{k-1} < t \le t_k, \ 0 \le x \le l, \ k = \overline{1,n},$$

$$\hat{u}^\tau(x,t) = u(x;k-1) + u_{\bar{t}}(x;k)(t - t_{k-1}), \quad \text{if } t_{k-1} < t \le t_k, \ 0 \le x \le l, \ k = \overline{1,n}$$

$$\hat{u}^\tau(x,t) = u(x;n), \quad \text{if } t \ge T, \ 0 \le x \le l.$$

$$u^\tau \in V_2(D), \quad \hat{u}^\tau \in W_2^{1,1}(D)$$

# Existence of the Optimal Control

$$D = \{(x,t):\ 0 < x < l,\ 0 < t \le T\}$$

$$a, b, c \in L_\infty(D),\ f \in L_2(D),$$

$$\phi \in W_2^1[0, s_0],\ \gamma, \chi \in W_2^{1,1}(D),\ \mu, \nu \in L_2[0,T],$$

$$\frac{\partial a}{\partial x} \in L_\infty(D), \quad \int_0^T \operatorname*{ess\ sup}_{0\le x\le l} \left|\frac{\partial a}{\partial t}\right| dt < +\infty. \tag{9}$$

## Theorem 2

*The Problem $\mathcal{I}$ has a solution, i.e.*

$$V_* = \{v \in V_R : \mathcal{J}(v) = \mathcal{J}_* \equiv \inf_{v\in V_R} \mathcal{J}(v)\} \ne \emptyset$$

# Convergence Theorem

## Theorem 3

*Sequence of discrete optimal control problems $\mathcal{I}_n$ approximates the optimal control problem $\mathcal{I}$ with respect to functional, i.e.*

$$\lim_{n\to+\infty} I_{n_*} = J_*, \tag{10}$$

*where*

$$I_{n_*} = \inf_{V_R^n} I_n([v]_n), \ n = 1, 2, ...$$

*If $[v]_{n_\epsilon} \in V_R^n$ is chosen such that*

$$I_{n_*} \le I_n([v]_{n_\epsilon}) \le I_{n_*} + \epsilon_n, \ \epsilon_n \downarrow 0,$$

*then the sequence $v_n = (s_n, g_n) = \mathcal{P}_n([v]_{n_\epsilon})$ converges to some element $v_* = (s_*, g_*) \in V_*$ weakly in $W_2^2[0,T] \times W_2^1[0,T]$, and strongly in $W_2^1[0,T] \times L_2[0,T]$. In particular $s_n$ converges to $s_*$ uniformly on $[0,T]$. Moreover, piecewise linear interpolation $\hat{u}^\tau$ of the discrete state vector $[u[v]_{n_\epsilon}]_n$ converges to the solution $u(x,t;v_*) \in W_2^{1,1}(\Omega_*)$ of the Neumann problem (2)-(5) weakly in $W_2^{1,1}(\Omega_*)$.*

# First Energy Estimate and its Consequences

## Theorem 4

*For all sufficiently small $\tau$ discrete state vector $[u([v]_n)]_n$ satisfies:*

$$\max_{0\le k\le n}\int_0^l u^2(x;k)\,dx+\tau\sum_{k=1}^n\int_0^l\Big|\frac{du(x;k)}{dx}\Big|^2dx\le$$

$$C\Big(\|\phi\|^2_{L_2(0,s_0)}+\|g^n\|^2_{L_2(0,T)}+\|f\|^2_{L_2(D)}+\|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)}$$

$$+\|\chi(s^n(t),t)\|^2_{L_2(0,T)}+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}}u^2(x;k)dx\Big),\qquad(11)$$

$$\max_{0\le k\le n}\int_0^l u^2(x;k)\,dx+\tau\sum_{k=0}^n\int_0^l\Big|\frac{du(x;k)}{dx}\Big|^2dx+\tau^2\sum_{k=1}^n\int_0^l u_{\bar t}^2(x;k)dx\le$$

$$C\Big(\|\phi\|^2_{W_2^1(0,s_0)}+\|g^n\|^2_{L_2(0,T)}+\|f\|^2_{L_2(D)}+\|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)}$$

$$+\|\chi(s^n(t),t)\|^2_{L_2(0,T)}+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}}u^2(x;k)dx\Big),\qquad(12)$$

# Weak Compactness in $V_2^{1,0}$

## Theorem 5

*Let $[v]_n \in V_R^n, n = 1, 2, ...$ be a sequence of discrete controls and the sequence $\{\mathcal{P}_n([v]_n\}$ converges strongly in $W_2^1[0,T] \times L_2[0,T]$ to $v = (s, g)$. Then the sequence $\{u^\tau\}$ converges as $\tau \to 0$ weakly in $W_2^{1,0}(\Omega)$ to weak solution $u \in V_2^{1,0}(\Omega)$ of the problem (2)-(5), i.e. to the solution of the integral identity (**??**). Moreover, $u$ satisfies the energy estimate*

$$\|u\|^2_{V_2^{1,0}(D)} \leq C\Big(\|\phi\|^2_{L_2(0,s_0)} + \|g\|^2_{L_2(0,T)} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,0}(D)} + \|\chi\|^2_{W_2^{1,0}(D)}\Big) \tag{13}$$

## Corollary 6

*For arbitrary $v = (s, g) \in V_R$ there exists a weak solution $u \in V_2^{1,0}(\Omega)$ of the problem (2)-(5) which satisfy the energy estimate (6):*

$$\|u\|^2_{V_2^{1,0}(D)} \leq C\Big(\|\phi\|^2_{L_2(0,s_0)} + \|g\|^2_{L_2(0,T)} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,0}(D)} + \|\chi\|^2_{W_2^{1,0}(D)}\Big)$$

# Second Energy Estimate and its Consequences

## Theorem 7

*For all sufficiently small $\tau$ discrete state vector $[u([v]_n)]_n$ satisfies the following stability estimation:*

$$\max_{1\le k\le n}\int_0^l \left|\frac{du(x;k)}{dx}\right|^2 dx+\tau\sum_{k=1}^n\int_0^l u_{\bar t}^2(x;k)\,dx\le C\Big[\,\|\phi\|^2_{W_2^1[0,l]}+$$
$$\|g^n\|^2_{W_2^{\frac14}[0,T]}+\|\gamma(s^n(t),t)(s^n)'(t)\|^2_{W_2^{\frac14}[0,T]}+\|\chi(s^n(t),t)\|^2_{W_2^{\frac14}[0,T]}+$$
$$+\|f\|^2_{L_2(D)}+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}}u^2(x;k)dx\Big] \qquad (14)$$

$W_2^{\frac14}[0,T]$ – Banach space of all elements of $L_2[0,T]$ with finite norm

$$\|u\|_{W_2^{\frac14}[0,T]}=\Big(\|u\|^2_{L_2[0,T]}+\int_0^T dt\int_0^T\frac{|u(t)-u(\tau)|^2}{|t-\tau|^{\frac32}}d\tau\Big)^{\frac12}$$

# Weak Compactness in $W_2^{1,1}$

## Theorem 8

*Let $[v]_n \in V_R^n, n = 1, 2, \ldots$ be a sequence of discrete controls and the sequence $\{\mathcal{P}_n([v]_n\}$ converges weakly in $W_2^2[0,T] \times W_2^1[0,T]$ to $v = (s, g)$. Then the sequence $\{\hat{u}^\tau\}$ converges as $\tau \to 0$ weakly in $W_2^{1,1}(\Omega)$ to weak solution $u \in W_2^{1,1}(\Omega)$ of the problem (2)-(5). Moreover, $u$ satisfies the energy estimate*

$$\|u\|^2_{W_2^{1,1}(D)} \leq C\Big(\|\phi\|^2_{W_2^1(0,s_0)} + \|g\|^2_{W_2^{\frac{1}{4}}} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,1}(D)} + \|\chi\|^2_{W_2^{1,1}(D)}\Big) \tag{15}$$

## Corollary 9

*For arbitrary $v = (s, g) \in V_R$ there exists a weak solution $u \in W_2^{1,1}(\Omega)$ of the problem (2)-(5) which satisfy the energy estimate*

$$\|u\|^2_{W_2^{1,1}(D)} \leq C\Big(\|\phi\|^2_{W_2^1(0,s_0)} + \|g\|^2_{W_2^{\frac{1}{4}}} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,1}(D)} + \|\chi\|^2_{W_2^{1,1}(D)}\Big)$$

# Sketch of the Proof of Theorem 8

Second energy estimate implies

$$\|\hat{u}^\tau\|^2_{W_2^{1,1}(D)} \leq C\Big(\|\phi\|^2_{W_2^1(0,s_0)}+\|g^n\|^2_{W_2^{\frac{1}{4}}}+\|f\|^2_{L_2(D)}+\|\gamma\|^2_{W_2^{1,1}(D)}+\|\chi\|^2_{W_2^{1,1}(D)}\Big)$$

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$$\|u\|^2_{W_2^{1,1}(D)} \leq C\Big(\|\phi\|^2_{W_2^1(0,s_0)} + \|g\|^2_{W_2^{\frac{1}{4}}} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,1}(D)} + \|\chi\|^2_{W_2^{1,1}(D)}\Big)$$

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$$\|\Delta u\|_{W_2^{1,1}(D)} \leq C \tag{16}$$

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# Proof of Existence Theorem

$$\int_0^T \int_0^{s(t)} \left\{ a\Delta u_x \Phi_x - b\Delta u_x \Phi - c\Delta u \Phi + \Delta u_t \Phi_x \right\} dx\, dt + \int_0^T (g_n - g)\,\Phi(0,t)$$
$$+ \int_0^T \left[\gamma(s_n(t),t)s_n'(t) - \gamma(s(t),t)s'(t) - \chi(s_n(t),t) + \chi(s(t),t)\right] \Phi(s(t),t)\, dt$$
$$+ \int_0^T \int_{s(t)}^{s_n(t)} \{a(u_n)_x\Phi_x - b(u_n)_x\Phi - cu_n\Phi + (u_n)_t\Phi + f\Phi\}\; dx\, dt$$
$$+ \int_0^T \left[\gamma(s_n(t),t)s_n'(t) - \chi(s_n(t),t)\right]\left[\Phi(s_n(t),t) - \Phi(s(t),t)\right]\; dt = 0 \tag{17}$$

# Proof of Existence Theorem

By using energy estimate (15), and continuity of traces $\gamma(s(t),t), \chi(s(t),t)$ of elements $\gamma, \chi \in W_2^{1,1}(D)$, strongly in $L_2[0,T]$ with respect to $s \in W_2^1[0,T]$, passing to the limit as $n \to +\infty$, from (17) it follows that the weak limit function $v$ satisfies the integral identity

$$\int_0^T \int_0^{s(t)} \{a v_x \Phi_x - b v_x \Phi + c v \Phi + v_t \Phi\} \, dx \, dt = 0 \tag{18}$$

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# Proof of Existence Theorem

From Sobolev trace theorem it follows that

$$\|u_n(0,t)-u(0,t)\|_{L^2[0,T]} \to 0, \ \|u_n(s(t),t)-u(s(t),t))\|_{L^2[0,T]} \to 0 \quad \text{as } n \to \infty,$$

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Hence, we have

$$\mathcal{J}(v) = \lim_{n\to\infty} \mathcal{J}(v_n) = \mathcal{J}_*$$

and $v$ is a solution of the Problem I. Theorem is proved.

## Lemma 10

*Let* $\mathcal{J}_*(\pm\epsilon) = \inf\limits_{V_{R\pm\epsilon}} \mathcal{J}(v)$, $\epsilon > 0$. *Then*

$$\lim_{\epsilon\to 0} \mathcal{J}_*(\epsilon) = \mathcal{J}_* = \lim_{\epsilon\to 0} \mathcal{J}_*(-\epsilon) \tag{19}$$

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$$s_{\epsilon'} \to s_* \text{ weakly in } W_2^2[0,T], \quad g_{\epsilon'} \to g_* \text{ weakly in } W_2^1[0,T], \quad \epsilon' \to 0$$

# Proof of Convergence Theorem

## Proof of Lemma 10, cont'd

The limit $v_* = (s_*, g_*) \in V_R$.

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The limit $v_* = (s_*, g_*) \in V_R$. By the weak continuity of $J$,

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## Lemma 11

*For arbitrary* $v = (s, g) \in V_R$,

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**Proof**: Let $v \in V_R$, $u = u(x, t; v)$, $Q_n(v) = [v]_n$, and $\big[u([v]_n)\big]_n$ be the corresponding discrete state vector. In theorem 8, it is proved that $\{\hat{u}^\tau\} \to u$ weakly in $W_2^{1,1}(\Omega)$.

$$\hat{u}^\tau(0, t) \to u(0, t), \quad \hat{u}^\tau\big(s(t), t\big) \to u\big(s(t), t\big) \text{ in } L_2[0, T]$$

## Lemma 11

*For arbitrary* $v = (s, g) \in V_R$,

$$\lim_{n\to\infty} \mathcal{I}_n(\mathcal{Q}_n(v)) = \mathcal{J}(v) \qquad (20)$$

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We now show $\{u^\tau(0, t)\}$, $\big\{u^\tau\big(s(t), t\big)\big\}$ converge strongly in $L_2[0, T]$.

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We now show $\{u^\tau(0,t)\}$, $\big\{u^\tau\big(s(t),t\big)\big\}$ converge strongly in $L_2[0,T]$. It is enough to show that $\{u^\tau\}$ and $\{\hat{u}^\tau\}$ are equivalent in the strong topology of $W_2^{1,0}(\Omega)$. In theorem 8, it is proved that they are equivalent in the strong topology of $L_2(D)$.

## Lemma 11

*For arbitrary* $v = (s, g) \in V_R$,

$$\lim_{n\to\infty} \mathcal{I}_n(\mathcal{Q}_n(v)) = \mathcal{J}(v) \tag{20}$$

**Proof**: Let $v \in V_R$, $u = u(x, t; v)$, $Q_n(v) = [v]_n$, and $\big[u([v]_n)\big]_n$ be the corresponding discrete state vector. In theorem 8, it is proved that $\{\hat{u}^\tau\} \to u$ weakly in $W_2^{1,1}(\Omega)$.

$$\hat{u}^\tau(0, t) \to u(0, t), \quad \hat{u}^\tau\big(s(t), t\big) \to u\big(s(t), t\big) \text{ in } L_2[0, T]$$

We now show $\{u^\tau(0, t)\}$, $\big\{u^\tau\big(s(t), t\big)\big\}$ converge strongly in $L_2[0, T]$. It is enough to show that $\{u^\tau\}$ and $\{\hat{u}^\tau\}$ are equivalent in the strong topology of $W_2^{1,0}(\Omega)$. In theorem 8, it is proved that they are equivalent in the strong topology of $L_2(D)$. It remains to show that $\{u^\tau_x\}$ and $\{\hat{u}^\tau_x\}$ are equivalent in the strong topology of $L_2(\Omega)$.

# Proof of Convergence Theorem

## Proof of Lemma 11, cont'd

$$\|u_x^\tau - \hat{u}_x^\tau\|^2_{L_2(\Omega)} \leq \frac{1}{3}\sum_{k=1}^{n}\tau^3 \int_0^{\min\left(s_{k-1};s_k\right)} \left(\frac{d\tilde{u}(x;k)}{dx}\right)^2_{\bar{t}} dx + \|u_x^\tau - \hat{u}_x^\tau\|^2_{L_2(\Gamma_n)}$$

# Proof of Convergence Theorem

## Proof of Lemma 11, cont'd

$$\|u_x^\tau - \hat{u}_x^\tau\|_{L_2(\Omega)}^2 \leq \frac{1}{3}\sum_{k=1}^{n} \tau^3 \int_0^{\min\left(s_{k-1};s_k\right)} \left(\frac{d\tilde{u}(x;k)}{dx}\right)_{\bar{t}}^2 dx + \|u_x^\tau - \hat{u}_x^\tau\|_{L_2(\Gamma_n)}^2$$

where $s_k = s^n(t_k)$, $s_n$ is the first component of $P_n([v]_n)$ and

$$\Gamma_n = \bigcup_{k=1}^{n} \left\{t_{k-1} < t \leq t_k;\ \min\left(s_{k-1}; s_k\right) < x < s(t)\right\}$$

# Proof of Convergence Theorem

## Proof of Lemma 11, cont'd

$$\|u_x^\tau - \hat{u}_x^\tau\|_{L_2(\Omega)}^2 \le \frac{1}{3}\sum_{k=1}^{n}\tau^3\int_0^{\min\left(s_{k-1};s_k\right)}\left(\frac{d\tilde{u}(x;k)}{dx}\right)_{\bar{t}}^2 dx + \|u_x^\tau - \hat{u}_x^\tau\|_{L_2(\Gamma_n)}^2$$

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Since $s^n$ converges uniformly on $[0,T]$, $|\Gamma_n| \to 0$.

# Proof of Convergence Theorem

## Proof of Lemma 11, cont'd

$$\|u_x^\tau - \hat{u}_x^\tau\|^2_{L_2(\Omega)} \leq \frac{1}{3}\sum_{k=1}^{n}\tau^3 \int_0^{\min\left(s_{k-1};s_k\right)} \left(\frac{d\tilde{u}(x;k)}{dx}\right)^2_{\bar{t}} dx + \|u_x^\tau - \hat{u}_x^\tau\|^2_{L_2(\Gamma_n)}$$

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Since $s^n$ converges uniformly on $[0,T]$, $|\Gamma_n| \to 0$. By theorems 5 and 8, the integrand is uniformly bounded, and hence the second term on the RHS $\to 0$.

# Proof of Convergence Theorem

Proof of Lemma 11, cont'd

$$\|u_x^\tau - \hat{u}_x^\tau\|^2_{L_2(\Omega)} \leq \frac{1}{3}\sum_{k=1}^{n} \tau^3 \int_0^{\min\left(s_{k-1};s_k\right)} \left(\frac{d\tilde{u}(x;k)}{dx}\right)^2_{\bar{t}} dx + \|u_x^\tau - \hat{u}_x^\tau\|^2_{L_2(\Gamma_n)}$$

where $s_k = s^n(t_k)$, $s_n$ is the first component of $P_n([v]_n)$ and

$$\Gamma_n = \bigcup_{k=1}^{n} \left\{t_{k-1} < t \leq t_k;\ \min\left(s_{k-1}; s_k\right) < x < s(t)\right\}$$

Since $s^n$ converges uniformly on $[0, T]$, $|\Gamma_n| \to 0$. By theorems 5 and 8, the integrand is uniformly bounded, and hence the second term on the RHS $\to 0$. The stability estimation implies the first term on the RHS $\to 0$, and hence $\{u_x^\tau\}$ $\{\hat{u}_x^\tau\}$ are equivalent in $L_2(\Omega)$.

Let $\nu^\tau(t) = \nu^k$, $\mu^\tau(t) = \mu^k$ if $t_{k-1} < t \leq t_k$, $k = 1, \ldots, n$.

# Proof of Convergence Theorem
## Proof of Lemma 11, cont'd

Let $\nu^\tau(t) = \nu^k$, $\mu^\tau(t) = \mu^k$ if $t_{k-1} < t \le t_k$, $k = 1, \dots, n$.

$$\left\|\nu^k - \nu\right\|_{L_2[0,T]} \to 0, \quad \left\|\mu^k - \mu\right\|_{L_2[0,T]} \to 0 \text{ as } \tau \to 0$$

Let $\nu^\tau(t) = \nu^k$, $\mu^\tau(t) = \mu^k$ if $t_{k-1} < t \le t_k$, $k = 1, \dots, n$.

$$\left\|\nu^k - \nu\right\|_{L_2[0,T]} \to 0, \quad \left\|\mu^k - \mu\right\|_{L_2[0,T]} \to 0 \text{ as } \tau \to 0$$

Estimate the first term in $\mathcal{I}_n\big(Q_n(v)\big)$ as

$$\beta_0 \tau \sum_{k=1}^{n} \left|u(0;k) - \nu^k\right|^2 = \beta_0 \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left|u(0;k) - \nu^k\right|^2 \, dt$$

Let $\nu^\tau(t) = \nu^k$, $\mu^\tau(t) = \mu^k$ if $t_{k-1} < t \le t_k$, $k = 1, \ldots, n$.

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$$= \beta_0 \int_0^T \left|u^\tau(0,t) - \nu^\tau\right|^2 \, dt$$

Let $\nu^\tau(t) = \nu^k$, $\mu^\tau(t) = \mu^k$ if $t_{k-1} < t \le t_k$, $k = 1, \dots, n$.

$$\left\|\nu^k - \nu\right\|_{L_2[0,T]} \to 0, \quad \left\|\mu^k - \mu\right\|_{L_2[0,T]} \to 0 \text{ as } \tau \to 0$$

Estimate the first term in $\mathcal{I}_n\big(Q_n(v)\big)$ as

$$\begin{aligned}\beta_0\tau\sum_{k=1}^{n}\left|u(0;k) - \nu^k\right|^2 &= \beta_0\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left|u(0;k) - \nu^k\right|^2\,dt\\ &= \beta_0\int_0^T\left|u^\tau(0,t) - \nu^\tau\right|^2\,dt\end{aligned}$$

$$\lim_{n\to\infty}\beta_0\tau\sum_{k=1}^{n}|u(0;k) - \nu^k|^2 = \beta_0\left\|u(0,t) - \nu(t)\right\|^2_{L_2[0,T]}$$

Estimate the second term in $\mathcal{I}_n\big(Q_n(v)\big)$ as

$$\beta_1\tau\sum_{k=1}^{n}\left|u(s_k;k)-\mu^k\right|^2=2\beta_1\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_{s(t)}^{s_k}\frac{\partial u^\tau}{\partial x}\big(u^\tau(s(t),t)-\mu^\tau(t)\big)\,dx\,dt+$$

$$+\beta_1\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left|u(s(t);k)-\mu^k\right|^2\,dt+\beta_1\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\int_{s(t)}^{s_k}\frac{\partial u^\tau}{\partial x}\,dx\right)^2\,dt$$

Estimate the second term in $\mathcal{I}_n\big(Q_n(v)\big)$ as

$$\beta_1\tau\sum_{k=1}^{n}\left|u(s_k;k)-\mu^k\right|^2 = 2\beta_1\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_{s(t)}^{s_k}\frac{\partial u^\tau}{\partial x}\big(u^\tau(s(t),t)-\mu^\tau(t)\big)\,dx\,dt+$$

$$+\beta_1\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left|u(s(t);k)-\mu^k\right|^2\,dt+\beta_1\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\int_{s(t)}^{s_k}\frac{\partial u^\tau}{\partial x}\,dx\right)^2\,dt$$

Set the three integrals on the RHS as $I_1$, $I_2$, and $I_3$.

Estimate the second term in $\mathcal{I}_n\big(Q_n(v)\big)$ as

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Set the three integrals on the RHS as $I_1$, $I_2$, and $I_3$.

$$\lim_{n\to\infty}I_2=\beta_1\int_0^T|u^\tau(s(t),t)-\mu^\tau(t)|^2\,dt=\beta_1\int_0^T|u(s(t),t)-\mu(t)|^2\,dt$$

Estimate the second term in $\mathcal{I}_n\big(Q_n(v)\big)$ as

$$\beta_1 \tau \sum_{k=1}^{n} \left| u(s_k;k) - \mu^k \right|^2 = 2\beta_1 \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_{s(t)}^{s_k} \frac{\partial u^\tau}{\partial x} \big( u^\tau(s(t),t) - \mu^\tau(t) \big)\, dx\, dt +$$

$$+ \beta_1 \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left| u(s(t);k) - \mu^k \right|^2 \, dt + \beta_1 \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left( \int_{s(t)}^{s_k} \frac{\partial u^\tau}{\partial x}\, dx \right)^2 \, dt$$

Set the three integrals on the RHS as $I_1$, $I_2$, and $I_3$.

$$\lim_{n\to\infty} I_2 = \beta_1 \int_0^T |u^\tau(s(t),t) - \mu^\tau(t)|^2 \, dt = \beta_1 \int_0^T |u(s(t),t) - \mu(t)|^2 \, dt$$

Since $\|(u^\tau)_x\|_{L_2(D)}$ and $\|u^\tau(s(t),t) - \mu^\tau\|_{L_2[0,T]}$ are unif. bounded and $\{s^n\}$ converges uniformly to $s$ on $[0,T]$,

$$\lim_{n\to\infty} I_1 = 0, \quad \lim_{n\to\infty} I_3 = 0$$

Hence

$$\lim_{\tau\to 0}\beta_1\tau\sum_{k=1}^{n}\left|u(s_k;k)-\mu^k\right|^2=\beta_1\int_0^T\left|u(s(t),t)-\mu(t)\right|^2\,dt$$

Lemma is proved.

## Lemma 12

*For arbitrary* $[v]_n \in V_R^n$

$$\lim_{n\to\infty}\Big(\mathcal{J}(\mathcal{P}_n([v]_n))-\mathcal{I}_n([v]_n)\Big)=0 \tag{21}$$