Optimal Control and Inverse Problems for PDEs.
Inverse Stefan Problem - Part 3

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May 16, 2014
Inverse Stefan Problem (ISP): Find the functions $u(x, t)$ and $s(t)$ and the boundary heat flux $g(t)$ satisfying conditions

\[
(a(x, t)u_x)_x + b(x, t)u_x + c(x, t)u - u_t = f(x, t), \quad \text{for } (x, t) \in \Omega
\]

\[
u(x, 0) = \phi(x), \quad 0 \leq x \leq s(0) = s_0
\]

\[
a(0, t)u_x(0, t) = g(t), \quad 0 \leq t \leq T
\]

\[
a(s(t), t)u_x(s(t), t) + \gamma(s(t), t)s'(t) = \chi(s(t), t), \quad 0 \leq t \leq T
\]

\[
u(s(t), t) = \mu(t), \quad 0 \leq t \leq T
\]

\[
u(0, t) = \nu(t), \quad \text{for } 0 \leq t \leq T
\]
Optimal Control Problem

\[ J(v) = \beta_0 \|u(0, t) - \nu(t)\|_{L^2[0,T]}^2 + \beta_1 \|u(s(t), t) - \mu(t)\|_{L^2[0,T]}^2 \quad (1) \]

\[ V_R = \left\{ v = (s, g) \in W_2^2[0,T] \times W_1^2[0,T] : \delta \leq s(t) \leq l, \right. \]  
\[ \left. s(0) = s_0, \max(\|s\|_{W_2^2}; \|g\|_{W_1^2}) \leq R \right\} \]

\[ (a(x,t)u_x)_x + b(x,t)u_x + c(x,t)u - u_t = f(x,t), \quad \text{for } (x,t) \in \Omega \quad (2) \]

\[ u(x,0) = \phi(x), \quad 0 \leq x \leq s(0) = s_0 \quad (3) \]

\[ a(0,t)u_x(0,t) = g(t), \quad 0 \leq t \leq T \quad (4) \]

\[ a(s(t), t)u_x(s(t), t) + \gamma(s(t), t)s'(t) = \chi(s(t), t), \quad 0 \leq t \leq T \quad (5) \]

\[ \Omega = \{(x,t) : 0 < x < s(t), 0 < t \leq T\} \]
\[ \omega_\tau = \{ t_j = j \cdot \tau, \ j = 0, 1, \ldots, n \} \]

\[ V_R^n = \{ [v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} : 0 < \delta \leq s_k \leq l, \] \[ \max([s]_n \| w_2^2; [g]_n \| w_1^2) \leq R^2 \} \]

\[ [s]_n = (s_0, s_1, \ldots, s_n) \in \mathbb{R}^{n+1}, [g]_n = (g_0, g_1, \ldots, g_n) \in \mathbb{R}^{n+1} \]

\[ \| [s]_n \| w_2^2 = \sum_{k=0}^{n-1} \tau s_k^2 + \sum_{k=1}^{n} \tau s_{t,k}^2 + \sum_{k=1}^{n-1} \tau s_{tt,k}^2, \] \[ \| [g]_n \| w_1^2 = \sum_{k=0}^{n-1} \tau g_k^2 + \sum_{k=1}^{n} \tau g_{t,k}^2. \]

\[ s_{\bar{t},k} = \frac{s_k - s_{k-1}}{\tau}, \ s_{t,k} = \frac{s_{k+1} - s_k}{\tau}, \ s_{\bar{tt},k} = \frac{s_{k+1} - 2s_k + s_{k-1}}{\tau^2}. \]
Definition 1

\[ [u([v]_{n})]_{n} = (u(x; 0), u(x; 1), ..., u(x; n)) \] is called discrete state vector if

1. \( u(x; 0) = \phi(x) \in W_{2}^{1}[0, s_{0}] \);
2. \( u(x; k) \in W_{2}^{1}[0, s_{k}] \) satisfy the integral identity

\[
\int_{0}^{s_{k}} \left( a_{k}(x) \frac{du(x; k)}{dx} \eta'(x) - b_{k} \frac{du(x; k)}{dx} \eta(x) - c_{k}(x) u(x; k) \eta(x) + f_{k}(x) \eta \right. \\
+ u_{t}(x; k) \eta(x) \bigg) \, dx + \left( (\gamma s^n (s^n)' \right)^{k} - \chi_{s^n}^{k} \right) \eta(s_{k}) + g_{k}^{n} \eta(0) = 0,
\]

\( \forall \eta \in W_{2}^{1}[0, s_{k}] \) \hspace{1cm} (6)

3. \( u(x; k) \in W_{2}^{1}[0, s_{k}] \) iteratively continued to \([0, l]\) as

\[ u(x; k) = u(2^n s_{k} - x; k), \quad 2^{n-1} s_{k} \leq x \leq 2^n s_{k}, n = 1, n_{k}, \]

\[ n_{k} \leq N = 1 + \log_{2} \left[ \frac{l}{\delta} \right] \] \hspace{1cm} (7)
Discrete Optimal Control Problem

\[ I_n([v]_n) = \beta_0 \tau \sum_{k=1}^{n} (u(0; k) - \nu_k)^2 + \beta_1 \tau \sum_{k=1}^{n} (u(s_k; k) - \mu_k)^2 \quad (8) \]

\[ V^n_R = \{ [v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} : 0 < \delta \leq s_k \leq l, \max(\|[s]_n\|_{w_2^2}^2; \|[g]_n\|_{w_2^1}^2) \leq R^2 \} \]

\[ [u([v]_n)]_n = (u(x; 0), u(x; 1), ..., u(x; n)) \] be a discrete state vector. Formulated discrete optimal control problem will be called Problem \( I_n \).

\[ u^\tau(x, t) = u(x; k), \quad \text{if } t_{k-1} < t \leq t_k, \; 0 \leq x \leq l, \; k = 1, n, \]

\[ \hat{u}^\tau(x, t) = u(x; k-1) + u_t(x; k)(t-t_{k-1}), \quad \text{if } t_{k-1} < t \leq t_k, \; 0 \leq x \leq l, \; k = 1, n \]

\[ \hat{u}^\tau(x, t) = u(x; n), \quad \text{if } t \geq T, \; 0 \leq x \leq l. \]

\[ u^\tau \in V_2(D), \quad \hat{u}^\tau \in W_2^{1,1}(D) \]
Existence of the Optimal Control

\[ D = \{(x, t) : 0 < x < l, \ 0 < t \leq T\} \]

\[ a, b, c \in L_\infty(D), \ f \in L_2(D), \]

\[ \phi \in W_2^1[0, s_0], \ \gamma, \chi \in W_2^{1,1}(D), \ \mu, \nu \in L_2[0, T], \]

\[ \frac{\partial a}{\partial x} \in L_\infty(D), \ \int_0^T \text{ess sup}_{0 \leq x \leq l} \left| \frac{\partial a}{\partial t} \right| \ dt < +\infty. \quad (9) \]

**Theorem 2**

*The Problem \( \mathcal{I} \) has a solution, i.e.*

\[ V_* = \{v \in V_R : J(v) = J_* \equiv \inf_{v \in V_R} J(v)\} \neq \emptyset \]
Convergence Theorem

**Theorem 3**

*Sequence of discrete optimal control problems* $I_n$ *approximates the optimal control problem* $I$ *with respect to functional, i.e.

\[
\lim_{n \to +\infty} I_{n*} = J_*,
\]

where

\[
I_{n*} = \inf_{V^n_R} I_n([v]_n), \ n = 1, 2, ... \]

If $[v]_{n_\epsilon} \in V^n_R$ is chosen such that

\[
I_{n*} \leq I_n([v]_{n_\epsilon}) \leq I_{n*} + \epsilon_n, \ \epsilon_n \downarrow 0,
\]

then the sequence $v_n = (s_n, g_n) = P_n([v]_{n_\epsilon})$ converges to some element $v_* = (s_*, g_*) \in V_*$ weakly in $W^2_2[0,T] \times W^1_2[0,T]$, and strongly in $W^1_2[0,T] \times L^2[0,T]$. In particular $s_n$ converges to $s_*$ uniformly on $[0,T]$. Moreover, piecewise linear interpolation $\hat{u}^\tau$ of the discrete state vector $[u[v]_{n_\epsilon}]_n$ converges to the solution $u(x,t;v_*) \in W^{1,1}_2(\Omega_*)$ of the Neumann problem (2)-(5) weakly in $W^{1,1}_2(\Omega_*)$. 
First Energy Estimate and its Consequences

**Theorem 4**

*For all sufficiently small τ discrete state vector* \([u([v]_n)]_n\) *satisfies:*

\[
\max_{0 \leq k \leq n} \int_0^l u^2(x; k) \, dx + \tau \sum_{k=1}^n \int_0^l \left| \frac{du(x; k)}{dx} \right|^2 \, dx \leq C \left( \|\phi\|_{L^2(0,s_0)}^2 + \|g^n\|_{L^2(0,T)}^2 + \|f\|_{L^2(D)}^2 + \|\gamma(s^n(t), t)(s^n)'(t)\|_{L^2(0,T)}^2 \right. \\
+ \|\chi(s^n(t), t)\|_{L^2(0,T)}^2 + \sum_{k=1}^{n-1} 1_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) \, dx \bigg), \quad (11)
\]

\[
\max_{0 \leq k \leq n} \int_0^l u^2(x; k) \, dx + \tau \sum_{k=1}^n \int_0^l \left| \frac{du(x; k)}{dx} \right|^2 \, dx + \tau^2 \sum_{k=1}^n \int_0^l u^2_t(x; k) \, dx \leq C \left( \|\phi\|_{W^1_2(0,s_0)}^2 + \|g^n\|_{L^2(0,T)}^2 + \|f\|_{L^2(D)}^2 + \|\gamma(s^n(t), t)(s^n)'(t)\|_{L^2(0,T)}^2 \right. \\
+ \|\chi(s^n(t), t)\|_{L^2(0,T)}^2 + \sum_{k=1}^{n-1} 1_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) \, dx \bigg), \quad (12)
\]

where \(C\) is independent of \(\tau\) and \(k\) be an indicator function of the positive semiaxis.
Weak Compactness in $V_{2}^{1,0}$

**Theorem 5**

Let $[v]_n \in V^n_R$, $n = 1, 2, \ldots$ be a sequence of discrete controls and the sequence $\{\mathcal{P}_n([v]_n)\}$ converges strongly in $W^1_2[0,T] \times L_2[0,T]$ to $v = (s, g)$. Then the sequence $\{u^\tau\}$ converges as $\tau \to 0$ weakly in $W^{1,0}_2(\Omega)$ to weak solution $u \in V^{1,0}_2(\Omega)$ of the problem (2)-(5), i.e. to the solution of the integral identity (??). Moreover, $u$ satisfies the energy estimate

$$
\|u\|_{V^{1,0}_2(D)}^2 \leq C \left( \|\phi\|_{L_2(0,s_0)}^2 + \|g\|_{L_2(0,T)}^2 + \|f\|_{L^2(D)}^2 + \|\gamma\|_{W^{1,0}_2(D)}^2 + \|\chi\|_{W^{1,0}_2(D)}^2 \right) 
$$

(13)

**Corollary 6**

For arbitrary $v = (s, g) \in V_R$ there exists a weak solution $u \in V^{1,0}_2(\Omega)$ of the problem (2)-(5) which satisfy the energy estimate (6):

$$
\|u\|_{V^{1,0}_2(D)}^2 \leq C \left( \|\phi\|_{L_2(0,s_0)}^2 + \|g\|_{L_2(0,T)}^2 + \|f\|_{L^2(D)}^2 + \|\gamma\|_{W^{1,0}_2(D)}^2 + \|\chi\|_{W^{1,0}_2(D)}^2 \right) 
$$
Second Energy Estimate and its Consequences

Theorem 7

For all sufficiently small \( \tau \) discrete state vector \([u([v]_n)]_n\) satisfies the following stability estimation:

\[
\max_{1 \leq k \leq n} \int_0^l \left| \frac{du(x; k)}{dx} \right|^2 \, dx + \tau \sum_{k=1}^n \int_0^l u_t^2(x; k) \, dx \leq C \left[ \| \phi \|^2_{W^1_2[0,l]} + \| g^n \|^2_{W^1_2[0,T]} + \| \gamma(s^n(t), t)(s^n)'(t) \|^2_{W^1_2[0,T]} + \| \chi(s^n(t), t) \|^2_{W^1_2[0,T]} + \| f \|^2_{L_2(D)} + \sum_{k=1}^{n-1} \left( 1 + (s_{k+1} - s_k) \right) \int_{s_k}^{s_{k+1}} u^2(x; k) \, dx \right] \quad (14)
\]

\( W^1_2[0,T] \) – Banach space of all elements of \( L_2[0,T] \) with finite norm

\[
\| u \|^2_{W^1_2[0,T]} = \left( \| u \|^2_{L_2[0,T]} + \int_0^T dt \int_0^T \frac{|u(t) - u(\tau)|^2}{|t - \tau|^{3/2}} \, d\tau \right)^{1/2}
\]
Weak Compactness in $W^{1,1}_2$

**Theorem 8**

Let $[v]_n \in V^n_R, n = 1, 2, \ldots$ be a sequence of discrete controls and the sequence $\{\mathcal{P}_n([v]_n)\}$ converges weakly in $W^2_2[0, T] \times W^1_2[0, T]$ to $v = (s, g)$. Then the sequence $\{\hat{u}^\tau\}$ converges as $\tau \to 0$ weakly in $W^{1,1}_2(\Omega)$ to weak solution $u \in W^{1,1}_2(\Omega)$ of the problem (2)-(5).

Moreover, $u$ satisfies the energy estimate

$$
\|u\|^2_{W^{1,1}_2(D)} \leq C\left(\|\phi\|^2_{W^1_2(0,s_0)} + \|g\|^2_{W^{1,2}_2} + \|f\|^2_{L^2(D)} + \|\gamma\|^2_{W^{1,1}_2(D)} + \|\chi\|^2_{W^{1,1}_2(D)}\right)
$$

(15)

**Corollary 9**

For arbitrary $v = (s, g) \in V_R$ there exists a weak solution $u \in W^{1,1}_2(\Omega)$ of the problem (2)-(5) which satisfy the energy estimate

$$
\|u\|^2_{W^{1,1}_2(D)} \leq C\left(\|\phi\|^2_{W^1_2(0,s_0)} + \|g\|^2_{W^{1,2}_2} + \|f\|^2_{L^2(D)} + \|\gamma\|^2_{W^{1,1}_2(D)} + \|\chi\|^2_{W^{1,1}_2(D)}\right)
$$
Sketch of the Proof of Theorem 8

Second energy estimate implies

\[ \| \hat{u}^\tau \|^2_{W^{1,1}_2(D)} \leq C \left( \| \phi \|^2_{W^{1}_2(0,s_0)} + \| g^n \|^2_{L^2(D)} + \| f \|^2_{L^2(D)} + \| \gamma \|^2_{W^{1,1}_2(D)} + \| \chi \|^2_{W^{1,1}_2(D)} \right) \]

\[ \Rightarrow \{ \hat{u}^\tau \} \text{ is weakly compact in } W^{1,1}_2(D) \Rightarrow \text{ strongly compact in } L^2(D) \]

Let \( u \) be a weak limit point of \( \{ \hat{u}^\tau \} \) in \( W^{1,1}_2(D) \) \( \Rightarrow u \) is a strong limit point of \( \{ \hat{u}^\tau \} \) in \( L^2(D) \)

\[ \| \hat{u}^\tau - u^\tau \|_{L^2(D)} = \frac{1}{3} \tau^3 \sum_{k=1}^{n} \int_{0}^{l} u_k^2(x; k) dx \rightarrow 0, \text{ as } \tau \rightarrow 0 \]

\[ \Rightarrow u \text{ is a strong limit of } \{ u^\tau \} \text{ in } L^2(D) \Rightarrow u \text{ is a unique weak solution of the problem (2)-(5) from } V^{1,0}_2(\Omega) \Rightarrow \{ \hat{u}^\tau \} \text{ converges weakly in } W^{1,1}_2(\Omega) \]

\[ \Rightarrow u \Rightarrow u \text{ is the weak solution of (2)-(5) from } W^{1,1}_2(\Omega) \Rightarrow \]

\[ \| u \|^2_{W^{1,1}_2(D)} \leq C \left( \| \phi \|^2_{W^{1}_2(0,s_0)} + \| g \|^2_{L^4(D)} + \| f \|^2_{L^2(D)} + \| \gamma \|^2_{W^{1,1}_2(D)} + \| \chi \|^2_{W^{1,1}_2(D)} \right) \]
Proof of Existence Theorem

Let \( \{v_n\} \in V_R \) be a minimizing sequence

\[
\lim_{n \to \infty} J(v_n) = J_\
\]

Sequence \( v_n = (s_n, g_n) \) is weakly precompact in \( W^2_2[0, T] \times W^1_2[0, T] \). Assume that the whole sequence \( v_n = (s_n, g_n) \) converge to some limit function \( v = (s, g) \in V_R \) weakly in \( W^2_2[0, T] \times W^1_2[0, T] \), and hence, strongly in \( W^1_2[0, T] \times L^2[0, T] \). Let \( u_n = u(x, t; v_n), u = u(x, t; v) \in W^{1,1}_2(D) \) are weak solutions of (2)-(5) in \( W^{1,1}_2(\Omega_n) \) and \( W^{1,1}_2(\Omega) \) respectively. By Corollary 9, both satisfy energy estimation (15) with \( g_n \) and \( g \) on the right-hand side respectively. Since \( v_n \in V_R, \|u_n\|_{W^{1,1}_2(D)} \) is uniformly bounded. Hence, the sequence \( \Delta u = u_n - u \) satisfies

\[
\|\Delta u\|_{W^{1,1}_2(D)} \leq C
\]

(16)

uniformly with respect to \( n \). Accordingly, \( \{\Delta u\} \) is weakly precompact in \( W^{1,1}_2(D) \). Without loss of generality assume that the whole sequence \( u_n - u \) converges weakly in \( W^{1,1}_2(D) \) to some function \( v \in W^{1,1}_2(D) \). Let us subtract integral identities for \( u_n \) and \( u \), by assuming that the fixed test function \( \Phi \) belongs to \( W^{1,1}_2(D) \).
\[
\int_0^T \int_0^{s(t)} \left\{ a \Delta u_x \Phi_x - b \Delta u_x \Phi - c \Delta u \Phi + \Delta u_t \Phi_x \right\} \, dx \, dt \n + \int_0^T \left( g_n - g \right) \Phi(0, t) \, dt 

+ \int_0^T \left[ \gamma(s_n(t), t)s_n'(t) - \gamma(s(t), t)s'(t) - \chi(s_n(t), t) + \chi(s(t), t) \right] \Phi(s(t), t) \, dt 

+ \int_0^T \int_{s_n(t)}^{s(t)} \left\{ a(u_n)_x \Phi_x - b(u_n)_x \Phi - cu_n \Phi + (u_n)_t \Phi + f \Phi \right\} \, dx \, dt 

+ \int_0^T \left[ \gamma(s_n(t), t)s_n'(t) - \chi(s_n(t), t) \right] \left[ \Phi(s_n(t), t) - \Phi(s(t), t) \right] \, dt = 0
\]  

(17)
Proof of Existence Theorem

By using energy estimate (15), and continuity of traces
\( \gamma(s(t),t), \chi(s(t),t) \) of elements \( \gamma, \chi \in W_{2,1}^1(D) \), strongly in \( L_2[0,T] \) with respect to \( s \in W_2^1[0,T] \), passing to the limit as \( n \to +\infty \), from (17) it follows that the weak limit function \( v \) satisfies the integral identity

\[
\int_0^T \int_0^{s(t)} \left\{ av_x \Phi_x - bv_x \Phi + cv \Phi + v_t \Phi \right\} \, dx \, dt = 0
\]

for arbitrary \( \Phi \in W_{2,1}^1(D) \). Since, any element \( \Phi \in W_{2,1}^1(\Omega) \) can be continued to \( D \) as element of \( W_{2,1}^1(D) \), (18) is valid for arbitrary \( \Phi \in W_{2,1}^1(\Omega) \). Hence, \( v \) is a weak solution from \( W_{2,1}^1(\Omega) \) of the problem (2)-(5) with \( f = g = \gamma = \chi = 0 \). From (15) and uniqueness it follows that \( v = 0 \). Thus \( u_n \) converges to \( u \) weakly in \( W_{2,1}^1(D) \).
From Sobolev trace theorem it follows that

$$\|u_n(0, t) - u(0, t)\|_{L^2[0, T]} \to 0, \quad \|u_n(s(t), t) - u(s(t), t)\|_{L^2[0, T]} \to 0 \quad \text{as } n \to \infty,$$

$$\|u_n(s_n(t), t) - u(s(t), t)\|_{L^2[0, T]} \leq \|u_n(s_n(t), t) - u_n(s(t), t)\|_{L^2[0, T]}$$

$$+ \|u_n(s(t), t) - u(s(t), t)\|_{L^2[0, T]} \to 0 \quad \text{as } n \to \infty.$$

Hence, we have

$$J(v) = \lim_{n \to \infty} J(v_n) = J_*$$

and $v$ is a solution of the Problem I. Theorem is proved.
Lemma 10

Let $J_*(\pm \epsilon) = \inf_{V_{R \pm \epsilon}} J(v)$, $\epsilon > 0$. Then

$$\lim_{\epsilon \to 0} J_*(\epsilon) = J_* = \lim_{\epsilon \to 0} J_*(-\epsilon)$$

(19)

Proof: For $0 < \epsilon_1 < \epsilon_2$,

$$J_*(\epsilon_2) \leq J_*(\epsilon_1) \leq J_* \leq J_*(-\epsilon_1) \leq J_*(-\epsilon_2)$$

$$\lim_{\epsilon \to 0} J_*(\epsilon) \leq J_* \quad \lim_{\epsilon \to 0} J_*(-\epsilon) \geq J_*$$ exist.

Choose $v_\epsilon \in V_{R+\epsilon}$ such that

$$\lim_{\epsilon \to 0} (J(v_\epsilon) - J_*(\epsilon)) = 0$$

$v_\epsilon = (s_\epsilon, g_\epsilon)$ is weakly precompact in $W^2_2[0,T] \times W^1_2[0,T]$ so, there exists $\epsilon'$ such that

$s_{\epsilon'} \to s_*$ weakly in $W^2_2[0,T], \quad g_{\epsilon'} \to g_*$ weakly in $W^1_2[0,T], \quad \epsilon' \to 0$
The limit $v_* = (s_*, g_*) \in V_R$. By the weak continuity of $J$,

$$\lim_{\epsilon' \to 0} J(v_{\epsilon'}) = J(v_*)$$

Since also

$$\lim_{\epsilon \to 0} \left( J(v_{\epsilon}) - J_*(\epsilon) \right) = 0,$$

it follows that

$$\lim_{\epsilon' \to 0} J(v_{\epsilon'}) = J_*$$

The first inequality in (19) follows. To prove the second inequality, fix $\epsilon_0 > 0$, $\tilde{v} = (\tilde{s}, \tilde{g} \in V_{R-\epsilon_0})$. Let $\alpha_k$ satisfy $0 < \alpha_k < 1$, $\lim_{k \to +\infty} \alpha_k = 0$ and set

$v_k = (s_k, g_k) := \alpha_k \tilde{v} + (1 - \alpha_k)v_*$

where $J(v_*) = J_*$. It follows that $J(v_k) \to J_*$. For fixed $k$, choose $\epsilon$ such that $0 < \epsilon < \epsilon_0 \alpha_k$. Then

$$J_*(-\epsilon) \leq J(v_k), \quad 0 < \epsilon < \epsilon_0 \alpha_k$$

$$\lim_{\epsilon \to 0} J_*(-\epsilon) \leq J(v_k) \quad \text{pass } k \to \infty \text{ to find } \lim_{\epsilon \to 0} J_*(-\epsilon) \leq J_*$$
Lemma 11

For arbitrary \( v = (s, g) \in V_R \),

\[
\lim_{n \to \infty} I_n(Q_n(v)) = J(v)
\]  

(20)

Proof: Let \( v \in V_R \), \( u = u(x, t; v) \), \( Q_n(v) = [v]_n \), and \( [u([v]_n)]_n \) be the corresponding discrete state vector. In theorem 8, it is proved that \( \{\hat{u}^\tau\} \to u \) weakly in \( W^{1,1}_2(\Omega) \).

\[
\hat{u}^\tau(0, t) \to u(0, t), \quad \hat{u}^\tau(s(t), t) \to u(s(t), t) \quad \text{in} \quad L_2[0, T]
\]

We now show \( \{u^\tau(0, t)\} \), \( \{u^\tau(s(t), t)\} \) converge strongly in \( L_2[0, T] \). It is enough to show that \( \{u^\tau\} \) and \( \{\hat{u}^\tau\} \) are equivalent in the strong topology of \( W^{1,0}_2(\Omega) \). In theorem 8, it is proved that they are equivalent in the strong topology of \( L_2(D) \). It remains to show that \( \{u^\tau_x\} \) and \( \{\hat{u}^\tau_x\} \) are equivalent in the strong topology of \( L_2(\Omega) \).
Proof of Convergence Theorem

Proof of Lemma 11, cont’d

\[ \| u_\tau^x - \hat{u}_\tau^x \|_{L^2(\Omega)}^2 \leq \frac{1}{3} \sum_{k=1}^{n} \tau^3 \int_0^\infty \min\left(s_{k-1}; s_k\right)^2 \left( \frac{d\tilde{u}(x; k)}{dx} \right)^2 dt + \| u_\tau^x - \hat{u}_\tau^x \|_{L^2(\Gamma_n)}^2 \]

where \( s_k = s^n(t_k) \), \( s_n \) is the first component of \( P_n([v]_n) \) and

\[ \Gamma_n = \bigcup_{k=1}^{n} \left\{ t_{k-1} < t \leq t_k; \ min\left(s_{k-1}; s_k\right) < x < s(t) \right\} \]

Since \( s^n \) converges uniformly on \([0, T]\), \( |\Gamma_n| \to 0 \). By theorems 5 and 8, the integrand is uniformly bounded, and hence the second term on the RHS \( \to 0 \). The stability estimation implies the first term on the RHS \( \to 0 \), and hence \( \{u_\tau^x\} \ \{\hat{u}_\tau^x\} \) are equivalent in \( L^2(\Omega) \).
Proof of Convergence Theorem

Proof of Lemma 11, cont’d

Let \( \nu^\tau(t) = \nu^k \), \( \mu^\tau(t) = \mu^k \) if \( t_{k-1} < t \leq t_k \), \( k = 1, \ldots, n \).

\[
\| \nu^k - \nu \|_{L^2[0,T]} \to 0, \quad \| \mu^k - \mu \|_{L^2[0,T]} \to 0 \quad \text{as} \quad \tau \to 0
\]

Estimate the first term in \( I_n(\nu) \) as

\[
\beta_0 \tau \sum_{k=1}^{n} |u(0; k) - \nu^k|^2 = \beta_0 \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} |u(0; k) - \nu^k|^2 \, dt
\]

\[
= \beta_0 \int_0^T |u^\tau(0, t) - \nu|^2 \, dt
\]

\[
\lim_{n \to \infty} \beta_0 \tau \sum_{k=1}^{n} |u(0; k) - \nu^k|^2 = \beta_0 \| u(0, t) - \nu(t) \|_{L^2[0,T]}^2
\]
Estimate the second term in $I_n(Q_n(v))$ as

$$
\beta_1 \tau \sum_{k=1}^{n} \left| u(s_k; k) - \mu^k \right|^2 = 2 \beta_1 \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_{s(t)}^{s_k} \frac{\partial u^\tau}{\partial x} (u^\tau(s(t), t) - \mu^\tau(t)) \, dx \, dt +
$$

$$
+ \beta_1 \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left| u(s(t); k) - \mu^k \right|^2 \, dt + \beta_1 \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left( \int_{s(t)}^{s_k} \frac{\partial u^\tau}{\partial x} \, dx \right)^2 \, dt
$$

Set the three integrals on the RHS as $I_1$, $I_2$, and $I_3$.

$$
\lim_{n\to\infty} I_2 = \beta_1 \int_{0}^{T} |u^\tau(s(t), t) - \mu^\tau(t)|^2 \, dt = \beta_1 \int_{0}^{T} |u(s(t), t) - \mu(t)|^2 \, dt
$$

Since $\|u^\tau \|_{L_2(D)}$ and $\|u^\tau(s(t), t) - \mu^\tau\|_{L_2[0,T]}$ are uniform bounded and $\{s^n\}$ converges uniformly to $s$ on $[0, T]$, $\lim_{n\to\infty} I_1 = 0$, $\lim_{n\to\infty} I_3 = 0$.
Proof of Convergence Theorem

Lemma 12

Hence

\[ \lim_{\tau \to 0} \beta_1 \tau \sum_{k=1}^{n} |u(s_k; k) - \mu_k|^2 = \beta_1 \int_0^T |u(s(t), t) - \mu(t)|^2 \, dt \]

Lemma is proved.

**Lemma 12**

For arbitrary \([v]_n \in V^n_R\)

\[ \lim_{n \to \infty} \left( \mathcal{J}(\mathcal{P}_n([v]_n)) - \mathcal{I}_n([v]_n) \right) = 0 \]  \hspace{1cm} (21)