

# Optimal Control and Inverse Problems for PDEs. Inverse Stefan Problem - Part 2

Ugur G. Abdulla

Florida Institute of Technology

May 20, 2014

**Inverse Stefan Problem (ISP):** Find the functions  $u(x, t)$  and  $s(t)$  and the boundary heat flux  $g(t)$  satisfying conditions

$$(a(x, t)u_x)_x + b(x, t)u_x + c(x, t)u - u_t = f(x, t), \quad \text{for } (x, t) \in \Omega$$

$$u(x, 0) = \phi(x), \quad 0 \leq x \leq s(0) = s_0$$

$$a(0, t)u_x(0, t) = g(t), \quad 0 \leq t \leq T$$

$$a(s(t), t)u_x(s(t), t) + \gamma(s(t), t)s'(t) = \chi(s(t), t), \quad 0 \leq t \leq T$$

$$u(s(t), t) = \mu(t), \quad 0 \leq t \leq T$$

$$u(0, t) = \nu(t), \quad \text{for } 0 \leq t \leq T$$

$$\mathcal{J}(v) = \beta_0 \|u(0, t) - \nu(t)\|_{L_2[0, T]}^2 + \beta_1 \|u(s(t), t) - \mu(t)\|_{L_2[0, T]}^2 \quad (1)$$

$$V_R = \left\{ v = (s, g) \in W_2^2[0, T] \times W_2^1[0, T] : \delta \leq s(t) \leq l, \right. \\ \left. s(0) = s_0, \max(\|s\|_{W_2^2}; \|g\|_{W_2^1}) \leq R \right\}$$

# Optimal Control Problem

$$\mathcal{J}(v) = \beta_0 \|u(0, t) - \nu(t)\|_{L_2[0, T]}^2 + \beta_1 \|u(s(t), t) - \mu(t)\|_{L_2[0, T]}^2 \quad (1)$$

$$V_R = \left\{ v = (s, g) \in W_2^2[0, T] \times W_2^1[0, T] : \delta \leq s(t) \leq l, \right. \\ \left. s(0) = s_0, \max(\|s\|_{W_2^2}; \|g\|_{W_2^1}) \leq R \right\}$$

$$(a(x, t)u_x)_x + b(x, t)u_x + c(x, t)u - u_t = f(x, t), \quad \text{for } (x, t) \in \Omega \quad (2)$$

$$u(x, 0) = \phi(x), \quad 0 \leq x \leq s(0) = s_0 \quad (3)$$

$$a(0, t)u_x(0, t) = g(t), \quad 0 \leq t \leq T \quad (4)$$

$$a(s(t), t)u_x(s(t), t) + \gamma(s(t), t)s'(t) = \chi(s(t), t), \quad 0 \leq t \leq T \quad (5)$$

$$\Omega = \{(x, t) : 0 < x < s(t), 0 < t \leq T\}$$

## Definition 1

The function  $u \in W_2^{1,1}(\Omega)$  is called a weak solution of the problem (2)-(5) if  $u(x, 0) = \phi(x) \in W_2^1[0, s_0]$  and

$$0 = \int_0^T \int_0^{s(t)} [au_x \Phi_x - bu_x \Phi - cu \Phi + u_t \Phi + f \Phi] dx dt + \int_0^T [\gamma(s(t), t)s'(t) - \chi(s(t), t)] \Phi(s(t), t) dt + \int_0^T g(t) \Phi(0, t) dt \quad (6)$$

for arbitrary  $\Phi \in W_2^{1,1}(\Omega)$

# $W_2^{1,1}(\Omega)$ -solution of the Neumann Problem

## Definition 1

The function  $u \in W_2^{1,1}(\Omega)$  is called a weak solution of the problem (2)-(5) if  $u(x, 0) = \phi(x) \in W_2^1[0, s_0]$  and

$$0 = \int_0^T \int_0^{s(t)} [au_x \Phi_x - bu_x \Phi - cu \Phi + u_t \Phi + f \Phi] dx dt + \int_0^T [\gamma(s(t), t)s'(t) - \chi(s(t), t)] \Phi(s(t), t) dt + \int_0^T g(t) \Phi(0, t) dt \quad (6)$$

for arbitrary  $\Phi \in W_2^{1,1}(\Omega)$

$W_2^{1,1}(\Omega)$  – Hilbert space of all elements of  $L_2(\Omega)$  whose weak derivatives  $\frac{\partial u}{\partial x}$ ,  $\frac{\partial u}{\partial t}$  belong to  $L_2(\Omega)$ , and scalar product is defined as

$$(u, v) = \int_{\Omega} \left( uv + \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial t} \frac{\partial v}{\partial t} \right) dx dt$$

# $V_2(\Omega)$ -solution of the Neumann Problem

## Definition 2

The function  $u \in V_2(\Omega)$  is called a weak solution of (2)-(5) if

$$\begin{aligned} 0 = & \int_0^T \int_0^{s(t)} [au_x \Phi_x - bu_x \Phi - cu \Phi - u \Phi_t + f \Phi] dx dt - \\ & - \int_0^{s_0} \phi(x) \Phi(x, 0) dx + \int_0^T g(t) \Phi(0, t) dt + \\ & + \int_0^T [\gamma(s(t), t) s'(t) - u(s(t), t) s'(t) - \chi(s(t), t)] \Phi(s(t), t) dt \quad (7) \end{aligned}$$

for arbitrary  $\Phi \in W_2^{1,1}(\Omega)$  such that  $\Phi|_{t=T} = 0$ .

# $V_2(\Omega)$ -solution of the Neumann Problem

## Definition 2

The function  $u \in V_2(\Omega)$  is called a weak solution of (2)-(5) if

$$\begin{aligned} 0 = & \int_0^T \int_0^{s(t)} [au_x \Phi_x - bu_x \Phi - cu \Phi - u \Phi_t + f \Phi] dx dt - \\ & - \int_0^{s_0} \phi(x) \Phi(x, 0) dx + \int_0^T g(t) \Phi(0, t) dt + \\ & + \int_0^T [\gamma(s(t), t) s'(t) - u(s(t), t) s'(t) - \chi(s(t), t)] \Phi(s(t), t) dt \quad (7) \end{aligned}$$

for arbitrary  $\Phi \in W_2^{1,1}(\Omega)$  such that  $\Phi|_{t=T} = 0$ .

$$\begin{aligned} \|u\|_{V_2(\Omega)} &= \left( \operatorname{ess\,sup}_{0 \leq t \leq T} \|u(x, t)\|_{L_2[0, s(t)]}^2 + \left\| \frac{\partial u}{\partial x} \right\|_{L_2(\Omega)}^2 \right)^{\frac{1}{2}} \\ \|u\|_{V_2^{1,0}(\Omega)} &= \left( \max_{0 \leq t \leq T} \|u(x, t)\|_{L_2(0, s(t))}^2 + \left\| \frac{\partial u}{\partial x} \right\|_{L_2(\Omega)}^2 \right)^{\frac{1}{2}} \end{aligned}$$

# $V_2(\Omega)$ -solution of the Neumann Problem

## Definition 2

The function  $u \in V_2(\Omega)$  is called a weak solution of (2)-(5) if

$$\begin{aligned} 0 = & \int_0^T \int_0^{s(t)} [au_x \Phi_x - bu_x \Phi - cu \Phi - u \Phi_t + f \Phi] dx dt - \\ & - \int_0^{s_0} \phi(x) \Phi(x, 0) dx + \int_0^T g(t) \Phi(0, t) dt + \\ & + \int_0^T [\gamma(s(t), t) s'(t) - u(s(t), t) s'(t) - \chi(s(t), t)] \Phi(s(t), t) dt \quad (7) \end{aligned}$$

for arbitrary  $\Phi \in W_2^{1,1}(\Omega)$  such that  $\Phi|_{t=T} = 0$ .

$$\begin{aligned} \|u\|_{V_2(\Omega)} &= \left( \operatorname{ess\,sup}_{0 \leq t \leq T} \|u(x, t)\|_{L_2[0, s(t)]}^2 + \left\| \frac{\partial u}{\partial x} \right\|_{L_2(\Omega)}^2 \right)^{\frac{1}{2}} \\ \|u\|_{V_2^{1,0}(\Omega)} &= \left( \max_{0 \leq t \leq T} \|u(x, t)\|_{L_2(0, s(t))}^2 + \left\| \frac{\partial u}{\partial x} \right\|_{L_2(\Omega)}^2 \right)^{\frac{1}{2}} \end{aligned}$$

Formulated optimal control problem will be called Problem  $\mathcal{I}$ .

# Discrete Optimal Control Problem

$$\omega_\tau = \{t_j = j \cdot \tau, j = 0, 1, \dots, n\}$$

# Discrete Optimal Control Problem

$$\omega_\tau = \{t_j = j \cdot \tau, j = 0, 1, \dots, n\}$$

$$V_R^n = \{[v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} : 0 < \delta \leq s_k \leq l, \\ \max(\|[s]_n\|_{w_2}^2; \|[g]_n\|_{w_1}^2) \leq R^2\}$$

$$\omega_\tau = \{t_j = j \cdot \tau, j = 0, 1, \dots, n\}$$

$$V_R^n = \{[v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} : 0 < \delta \leq s_k \leq l, \\ \max(\|[s]_n\|_{w_2}^2; \|[g]_n\|_{w_1}^2) \leq R^2\}$$

$$[s]_n = (s_0, s_1, \dots, s_n) \in \mathbb{R}^{n+1}, [g]_n = (g_0, g_1, \dots, g_n) \in \mathbb{R}^{n+1}$$

# Discrete Optimal Control Problem

$$\omega_\tau = \{t_j = j \cdot \tau, j = 0, 1, \dots, n\}$$

$$V_R^n = \{[v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} : 0 < \delta \leq s_k \leq l, \\ \max(\|[s]_n\|_{w_2}^2, \|[g]_n\|_{w_1}^2) \leq R^2\}$$

$$[s]_n = (s_0, s_1, \dots, s_n) \in \mathbb{R}^{n+1}, [g]_n = (g_0, g_1, \dots, g_n) \in \mathbb{R}^{n+1}$$

$$\|[s]_n\|_{w_2}^2 = \sum_{k=0}^{n-1} \tau s_k^2 + \sum_{k=1}^n \tau s_{\bar{t},k}^2 + \sum_{k=1}^{n-1} \tau s_{\bar{t}\bar{t},k}^2, \|[g]_n\|_{w_1}^2 = \sum_{k=0}^{n-1} \tau g_k^2 + \sum_{k=1}^n \tau g_{\bar{t},k}^2.$$

$$s_{\bar{t},k} = \frac{s_k - s_{k-1}}{\tau}, s_{t,k} = \frac{s_{k+1} - s_k}{\tau}, s_{\bar{t}\bar{t},k} = \frac{s_{k+1} - 2s_k + s_{k-1}}{\tau^2}.$$

# Approximation and Interpolation Mappings

Introduce two mappings  $\mathcal{Q}_n$  and  $\mathcal{P}_n$  between continuous and discrete control sets:

$$\mathcal{Q}_n(v) = [v]_n = ([s]_n, [g]_n), \quad \text{for } v \in V_R$$

where  $s_k = s(t_k), g_k = g(t_k), k = 0, 1, \dots, n$ .

# Approximation and Interpolation Mappings

Introduce two mappings  $\mathcal{Q}_n$  and  $\mathcal{P}_n$  between continuous and discrete control sets:

$$\mathcal{Q}_n(v) = [v]_n = ([s]_n, [g]_n), \quad \text{for } v \in V_R$$

where  $s_k = s(t_k)$ ,  $g_k = g(t_k)$ ,  $k = 0, 1, \dots, n$ .

$$\mathcal{P}_n([v]_n) = v^n = (s^n, g^n) \in W_2^2[0, T] \times W_2^1[0, T] \quad \text{for } [v]_n \in V_R^n,$$

$$s^n(t) = \begin{cases} s_0 + \frac{t^2}{2\tau} s_{\bar{t},1} & 0 \leq t \leq \tau, \\ s_{k-1} + (t - t_{k-1} - \frac{\tau}{2}) s_{\bar{t},k-1} + \frac{1}{2}(t - t_{k-1})^2 s_{\bar{t}t,k-1} & t_{k-1} \leq t \leq t_k. \end{cases}$$

$$g^n(t) = g_{k-1} + \frac{g_k - g_{k-1}}{\tau}(t - t_{k-1}), \quad t_{k-1} \leq t \leq t_k, k = \overline{1, n}.$$

$$d_k(x) = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} d(x, t) dt, \quad h_k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} h(t) dt,$$

# Steklov Averages

$$d_k(x) = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} d(x, t) dt, \quad h_k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} h(t) dt,$$

$d$  stands for any of the functions  $a, b, c, f$ , and  $h$  stands for any of the functions  $\nu, \mu, g, g^n$ .

$$d_k(x) = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} d(x, t) dt, \quad h_k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} h(t) dt,$$

$d$  stands for any of the functions  $a, b, c, f$ , and  $h$  stands for any of the functions  $\nu, \mu, g, g^n$ .

Given  $v = (s, g) \in V_R$  define Steklov averages of traces

$$\chi_s^k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \chi(s(t), t) dt, \quad (\gamma_s s')^k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \gamma(s(t), t) s'(t) dt.$$

$$d_k(x) = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} d(x, t) dt, \quad h_k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} h(t) dt,$$

$d$  stands for any of the functions  $a, b, c, f$ , and  $h$  stands for any of the functions  $\nu, \mu, g, g^n$ .

Given  $v = (s, g) \in V_R$  define Steklov averages of traces

$$\chi_s^k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \chi(s(t), t) dt, \quad (\gamma_s s')^k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \gamma(s(t), t) s'(t) dt.$$

Given  $[v]_n = ([s]_n, [g]_n) \in V_R^n$  define Steklov averages of traces

$$\chi_{s^n}^k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \chi(s^n(t), t) dt, \quad (\gamma_{s^n} (s^n)')^k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \gamma(s^n(t), t) (s^n)'(t) dt.$$

## Definition 3

$[u([v]_n)]_n = (u(x; 0), u(x; 1), \dots, u(x; n))$  is called discrete state vector if

1.  $u(x; 0) = \phi(x) \in W_2^1[0, s_0]$ ;

## Definition 3

$[u([v]_n)]_n = (u(x; 0), u(x; 1), \dots, u(x; n))$  is called discrete state vector if

1.  $u(x; 0) = \phi(x) \in W_2^1[0, s_0]$ ;
2.  $u(x; k) \in W_2^1[0, s_k]$  satisfy the integral identity

$$\int_0^{s_k} \left( a_k(x) \frac{du(x; k)}{dx} \eta'(x) - b_k \frac{du(x; k)}{dx} \eta(x) - c_k(x) u(x; k) \eta(x) + f_k(x) \eta + u_{\bar{t}}(x; k) \eta(x) \right) dx + \left( (\gamma_{s^n} (s^n)')^k - \chi_{s^n}^k \right) \eta(s_k) + g_k^n \eta(0) = 0, \\ \forall \eta \in W_2^1[0, s_k] \quad (8)$$

## Definition 3

$[u([v]_n)]_n = (u(x; 0), u(x; 1), \dots, u(x; n))$  is called discrete state vector if

1.  $u(x; 0) = \phi(x) \in W_2^1[0, s_0]$ ;
2.  $u(x; k) \in W_2^1[0, s_k]$  satisfy the integral identity

$$\int_0^{s_k} \left( a_k(x) \frac{du(x; k)}{dx} \eta'(x) - b_k \frac{du(x; k)}{dx} \eta(x) - c_k(x) u(x; k) \eta(x) + f_k(x) \eta + u_{\bar{t}}(x; k) \eta(x) \right) dx + \left( (\gamma_{s^n} (s^n)')^k - \chi_{s^n}^k \right) \eta(s_k) + g_k^n \eta(0) = 0, \\ \forall \eta \in W_2^1[0, s_k] \quad (8)$$

3.  $u(x; k) \in W_2^1[0, s_k]$  iteratively continued to  $[0, l]$  as

$$u(x; k) = u(2^n s_k - x; k), \quad 2^{n-1} s_k \leq x \leq 2^n s_k, \quad n = \overline{1, n_k},$$

$$n_k \leq N = 1 + \log_2 \left[ \frac{l}{\delta} \right] \quad (9)$$

# Discrete Optimal Control Problem

$$\mathcal{I}_n([v]_n) = \beta_0 \tau \sum_{k=1}^n \left( u(0; k) - \nu_k \right)^2 + \beta_1 \tau \sum_{k=1}^n \left( u(s_k; k) - \mu_k \right)^2 \quad (10)$$

# Discrete Optimal Control Problem

$$\mathcal{I}_n([v]_n) = \beta_0 \tau \sum_{k=1}^n \left( u(0; k) - \nu_k \right)^2 + \beta_1 \tau \sum_{k=1}^n \left( u(s_k; k) - \mu_k \right)^2 \quad (10)$$

$$V_R^n = \{ [v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} : 0 < \delta \leq s_k \leq l, \\ \max(\|[s]_n\|_{w_2}^2; \|[g]_n\|_{w_1}^2) \leq R^2 \}$$

# Discrete Optimal Control Problem

$$\mathcal{I}_n([v]_n) = \beta_0 \tau \sum_{k=1}^n \left( u(0; k) - \nu_k \right)^2 + \beta_1 \tau \sum_{k=1}^n \left( u(s_k; k) - \mu_k \right)^2 \quad (10)$$

$$V_R^n = \{ [v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} : 0 < \delta \leq s_k \leq l, \\ \max(\| [s]_n \|_{w_2^2}, \| [g]_n \|_{w_1^2}) \leq R^2 \}$$

$[u([v]_n)]_n = (u(x; 0), u(x; 1), \dots, u(x; n))$  be a discrete state vector.

Formulated discrete optimal control problem will be called Problem  $\mathcal{I}_n$ .

# Discrete Optimal Control Problem

$$\mathcal{I}_n([v]_n) = \beta_0 \tau \sum_{k=1}^n \left( u(0; k) - \nu_k \right)^2 + \beta_1 \tau \sum_{k=1}^n \left( u(s_k; k) - \mu_k \right)^2 \quad (10)$$

$$V_R^n = \{ [v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} : 0 < \delta \leq s_k \leq l, \\ \max(\| [s]_n \|_{w_2^2}^2; \| [g]_n \|_{w_1^2}^2) \leq R^2 \}$$

$[u([v]_n)]_n = (u(x; 0), u(x; 1), \dots, u(x; n))$  be a discrete state vector.

Formulated discrete optimal control problem will be called Problem  $\mathcal{I}_n$ .

$$u^\tau(x, t) = u(x; k), \quad \text{if } t_{k-1} < t \leq t_k, \quad 0 \leq x \leq l, \quad k = \overline{1, n},$$

$$\hat{u}^\tau(x, t) = u(x; k-1) + u_{\bar{t}}(x; k)(t - t_{k-1}), \quad \text{if } t_{k-1} < t \leq t_k, \quad 0 \leq x \leq l, \quad k = \overline{1, n}$$

$$\hat{u}^\tau(x, t) = u(x; n), \quad \text{if } t \geq T, \quad 0 \leq x \leq l.$$

$$u^\tau \in V_2(D), \quad \hat{u}^\tau \in W_2^{1,1}(D)$$

$$\begin{aligned} D &= \{(x, t) : 0 < x < l, 0 < t \leq T\} \\ a, b, c &\in L_\infty(D), f \in L_2(D), \\ \phi &\in W_2^1[0, s_0], \gamma, \chi \in W_2^{1,1}(D), \mu, \nu \in L_2[0, T], \\ \frac{\partial a}{\partial x} &\in L_\infty(D), \int_0^T \operatorname{ess\,sup}_{0 \leq x \leq l} \left| \frac{\partial a}{\partial t} \right| dt < +\infty. \end{aligned} \quad (11)$$

$$\begin{aligned} D &= \{(x, t) : 0 < x < l, 0 < t \leq T\} \\ a, b, c &\in L_\infty(D), f \in L_2(D), \\ \phi &\in W_2^1[0, s_0], \gamma, \chi \in W_2^{1,1}(D), \mu, \nu \in L_2[0, T], \\ \frac{\partial a}{\partial x} &\in L_\infty(D), \int_0^T \operatorname{ess\,sup}_{0 \leq x \leq l} \left| \frac{\partial a}{\partial t} \right| dt < +\infty. \end{aligned} \quad (11)$$

## Theorem 4

*The Problem  $\mathcal{I}$  has a solution, i.e.*

$$V_* = \{v \in V_R : \mathcal{J}(v) = \mathcal{J}_* \equiv \inf_{v \in V_R} \mathcal{J}(v)\} \neq \emptyset$$

## Theorem 5

*Sequence of discrete optimal control problems  $\mathcal{I}_n$  approximates the optimal control problem  $\mathcal{I}$  with respect to functional, i.e.*

$$\lim_{n \rightarrow +\infty} I_{n_*} = J_*, \quad (12)$$

where

$$I_{n_*} = \inf_{V_R^n} I_n([v]_n), \quad n = 1, 2, \dots$$

If  $[v]_{n_\epsilon} \in V_R^{n_\epsilon}$  is chosen such that

$$I_{n_*} \leq I_{n_\epsilon}([v]_{n_\epsilon}) \leq I_{n_*} + \epsilon_n, \quad \epsilon_n \downarrow 0,$$

then the sequence  $v_n = (s_n, g_n) = \mathcal{P}_n([v]_{n_\epsilon})$  converges to some element  $v_* = (s_*, g_*) \in V_*$  weakly in  $W_2^2[0, T] \times W_2^1[0, T]$ , and strongly in  $W_2^1[0, T] \times L_2[0, T]$ . In particular  $s_n$  converges to  $s_*$  uniformly on  $[0, T]$ . Moreover, piecewise linear interpolation  $\hat{u}^\tau$  of the discrete state vector  $[u[v]_{n_\epsilon}]_n$  converges to the solution  $u(x, t; v_*) \in W_2^{1,1}(\Omega_*)$  of the Neumann problem (2)-(5) weakly in  $W_2^{1,1}(\Omega_*)$ .

## Lemma 6

*For sufficiently small time step  $\tau$ , there exists a unique discrete state vector  $[u([v]_n)]_n$  for arbitrary discrete control vector  $[v]_n \in V_R^n$ .*

## Lemma 7

*For arbitrary sufficiently small  $\epsilon > 0$  there exists  $n_\epsilon$  such that*

$$Q_n(v) \in V_R^n, \quad \text{for all } v \in V_{R-\epsilon} \quad \text{and } n > n_\epsilon. \quad (13)$$

## Lemma 6

*For sufficiently small time step  $\tau$ , there exists a unique discrete state vector  $[u([v]_n)]_n$  for arbitrary discrete control vector  $[v]_n \in V_R^n$ .*

## Lemma 7

*For arbitrary sufficiently small  $\epsilon > 0$  there exists  $n_\epsilon$  such that*

$$\mathcal{Q}_n(v) \in V_R^n, \quad \text{for all } v \in V_{R-\epsilon} \quad \text{and } n > n_\epsilon. \quad (13)$$

$$\mathcal{P}_n([v]_n) \in V_{R+\epsilon}, \quad \text{for all } [v]_n \in V_R^n \quad \text{and } n > n_\epsilon. \quad (14)$$

## Lemma 6

For sufficiently small time step  $\tau$ , there exists a unique discrete state vector  $[u([v]_n)]_n$  for arbitrary discrete control vector  $[v]_n \in V_R^n$ .

## Lemma 7

For arbitrary sufficiently small  $\epsilon > 0$  there exists  $n_\epsilon$  such that

$$\mathcal{Q}_n(v) \in V_R^n, \quad \text{for all } v \in V_{R-\epsilon} \quad \text{and } n > n_\epsilon. \quad (13)$$

$$\mathcal{P}_n([v]_n) \in V_{R+\epsilon}, \quad \text{for all } [v]_n \in V_R^n \quad \text{and } n > n_\epsilon. \quad (14)$$

## Corollary 8

Let either  $[v]_n \in V_R^n$  or  $[v]_n = \mathcal{Q}_n(v)$  for  $v \in V_R$ . Then

$$|s_k - s_{k-1}| \leq C\tau, \quad k = 1, 2, \dots, n \quad (15)$$

where  $C$  is independent of  $n$ .

## Lemma 9

(Tikhonov, Vasil'ev 1980) Sequence  $I_n$  approximates the continuous optimal control problem  $I$  if and only if the following conditions are satisfied:

1. For arbitrary sufficiently small  $\epsilon > 0$  there exists a number  $N_1 = N_1(\epsilon)$  such that  $\mathcal{Q}_N(v) \in V_R^n$  for all  $v \in V_{R-\epsilon}$  and  $N \geq N_1$ ; and for any fixed  $\epsilon > 0$  and for all  $v \in V_{R-\epsilon}$  the following inequality is satisfied:

$$\limsup_{N \rightarrow \infty} \left( I_N(\mathcal{Q}_N(v)) - J(v) \right) \leq 0. \quad (16)$$

2. For arbitrary sufficiently small  $\epsilon > 0$  there exists a number  $N_2 = N_2(\epsilon)$  such that  $\mathcal{P}_N([v]_N) \in V_{R+\epsilon}$  for all  $[v]_N \in V_R^N$  and  $N \geq N_2$ ; and for all  $[v]_N \in V_R^N$ ,  $N \geq 1$  the following inequality is satisfied:

$$\limsup_{N \rightarrow \infty} \left( J(\mathcal{P}_N([v]_N)) - I_N([v]_N) \right) \leq 0. \quad (17)$$

3.  $\limsup_{\epsilon \rightarrow 0} J_*(\epsilon) \geq J_*$ ,  $\liminf_{\epsilon \rightarrow 0} J_*(-\epsilon) \leq J_*$ , where

$$J_*(\pm\epsilon) = \inf_{V_{R\pm\epsilon}} J(u)$$

# First Energy Estimate and its Consequences

## Theorem 10

For all sufficiently small  $\tau$  discrete state vector  $[u([v]_n)]_n$  satisfies:

$$\begin{aligned} & \max_{0 \leq k \leq n} \int_0^l u^2(x; k) dx + \tau \sum_{k=1}^n \int_0^l \left| \frac{du(x; k)}{dx} \right|^2 dx \leq \\ & C \left( \|\phi\|_{L_2(0, s_0)}^2 + \|g^n\|_{L_2(0, T)}^2 + \|f\|_{L_2(D)}^2 + \|\gamma(s^n(t), t)(s^n)'(t)\|_{L_2(0, T)}^2 \right. \\ & \left. + \|\chi(s^n(t), t)\|_{L_2(0, T)}^2 + \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx \right), \quad (18) \end{aligned}$$

# First Energy Estimate and its Consequences

## Theorem 10

For all sufficiently small  $\tau$  discrete state vector  $[u([v]_n)]_n$  satisfies:

$$\max_{0 \leq k \leq n} \int_0^l u^2(x; k) dx + \tau \sum_{k=1}^n \int_0^l \left| \frac{du(x; k)}{dx} \right|^2 dx \leq$$

$$C \left( \|\phi\|_{L_2(0, s_0)}^2 + \|g^n\|_{L_2(0, T)}^2 + \|f\|_{L_2(D)}^2 + \|\gamma(s^n(t), t)(s^n)'(t)\|_{L_2(0, T)}^2 \right. \\ \left. + \|\chi(s^n(t), t)\|_{L_2(0, T)}^2 + \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx \right), \quad (18)$$

$$\max_{0 \leq k \leq n} \int_0^l u^2(x; k) dx + \tau \sum_{k=0}^n \int_0^l \left| \frac{du(x; k)}{dx} \right|^2 dx + \tau^2 \sum_{k=1}^n \int_0^l u_t^2(x; k) dx \leq$$

$$C \left( \|\phi\|_{W_2^1(0, s_0)}^2 + \|g^n\|_{L_2(0, T)}^2 + \|f\|_{L_2(D)}^2 + \|\gamma(s^n(t), t)(s^n)'(t)\|_{L_2(0, T)}^2 \right. \\ \left. + \|\chi(s^n(t), t)\|_{L_2(0, T)}^2 + \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx \right), \quad (19)$$

## Theorem 11

Let  $[v]_n \in V_R^n$ ,  $n = 1, 2, \dots$  be a sequence of discrete controls and the sequence  $\{\mathcal{P}_n([v]_n)\}$  converges strongly in  $W_2^1[0, T] \times L_2[0, T]$  to  $v = (s, g)$ . Then the sequence  $\{u^\tau\}$  converges as  $\tau \rightarrow 0$  weakly in  $W_2^{1,0}(\Omega)$  to weak solution  $u \in V_2^{1,0}(\Omega)$  of the problem (2)-(5), i.e. to the solution of the integral identity (7). Moreover,  $u$  satisfies the energy estimate

$$\|u\|_{V_2^{1,0}(D)}^2 \leq C \left( \|\phi\|_{L_2(0,s_0)}^2 + \|g\|_{L_2(0,T)}^2 + \|f\|_{L_2(D)}^2 + \|\gamma\|_{W_2^{1,0}(D)}^2 + \|\chi\|_{W_2^{1,0}(D)}^2 \right) \quad (20)$$

## Theorem 11

Let  $[v]_n \in V_R^n$ ,  $n = 1, 2, \dots$  be a sequence of discrete controls and the sequence  $\{\mathcal{P}_n([v]_n)\}$  converges strongly in  $W_2^1[0, T] \times L_2[0, T]$  to  $v = (s, g)$ . Then the sequence  $\{u^\tau\}$  converges as  $\tau \rightarrow 0$  weakly in  $W_2^{1,0}(\Omega)$  to weak solution  $u \in V_2^{1,0}(\Omega)$  of the problem (2)-(5), i.e. to the solution of the integral identity (7). Moreover,  $u$  satisfies the energy estimate

$$\|u\|_{V_2^{1,0}(D)}^2 \leq C \left( \|\phi\|_{L_2(0,s_0)}^2 + \|g\|_{L_2(0,T)}^2 + \|f\|_{L_2(D)}^2 + \|\gamma\|_{W_2^{1,0}(D)}^2 + \|\chi\|_{W_2^{1,0}(D)}^2 \right) \quad (20)$$

Proof.

$$\tilde{s}^n(t) = s_{k-1} + \frac{s_k - s_{k-1}}{\tau} (t - t_{k-1}), \quad t_{k-1} \leq t \leq t_k, \quad k = \overline{1, n};$$

$$\tilde{s}^n(t) \equiv s_n, \quad t \geq T; \quad \tilde{s}_1^n(t) = \tilde{s}^n(t + \tau), \quad 0 \leq t \leq T.$$

$$\sup_n \|\tilde{s}_1^n\|_{W_2^1[0, T]} < C_* \quad (21)$$

where  $C_*$  is independent of  $n$ .

# Proof of Weak Compactness in $V_2^{1,0}$

$$\begin{aligned} \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx &= \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{t_k}^{t_{k+1}} (\tilde{s}^n)'(t) \times \\ u^2(\tilde{s}^n(t); k) dt &= \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{t_k}^{t_{k+1}} (\tilde{s}^n)'(t) \left( u^\tau(\tilde{s}^n(t), t - \tau) \right)^2 dt = \\ \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{t_{k-1}}^{t_k} (\tilde{s}_1^n)'(t) \left( u^\tau(\tilde{s}_1^n(t), t) \right)^2 dt. \quad (22) \end{aligned}$$

# Proof of Weak Compactness in $V_2^{1,0}$

$$\begin{aligned}
 \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx &= \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{t_k}^{t_{k+1}} (\tilde{s}^n)'(t) \times \\
 u^2(\tilde{s}^n(t); k) dt &= \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{t_k}^{t_{k+1}} (\tilde{s}^n)'(t) \left( u^\tau(\tilde{s}^n(t), t - \tau) \right)^2 dt = \\
 \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{t_{k-1}}^{t_k} (\tilde{s}_1^n)'(t) \left( u^\tau(\tilde{s}_1^n(t), t) \right)^2 dt. &\quad (22)
 \end{aligned}$$

$$\left| \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx \right| \leq \|(\tilde{s}_1^n)'\|_{L_2[0, T]} \|u^\tau(\tilde{s}_1^n(t), t)\|_{L_4[0, T]}^2. \quad (23)$$

# Proof of Weak Compactness in $V_2^{1,0}$

$$\begin{aligned}
 \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx &= \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{t_k}^{t_{k+1}} (\tilde{s}^n)'(t) \times \\
 u^2(\tilde{s}^n(t); k) dt &= \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{t_k}^{t_{k+1}} (\tilde{s}^n)'(t) \left( u^\tau(\tilde{s}^n(t), t - \tau) \right)^2 dt = \\
 \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{t_{k-1}}^{t_k} (\tilde{s}_1^n)'(t) \left( u^\tau(\tilde{s}_1^n(t), t) \right)^2 dt. &\quad (22)
 \end{aligned}$$

$$\left| \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx \right| \leq \|(\tilde{s}_1^n)'\|_{L_2[0,T]} \|u^\tau(\tilde{s}_1^n(t), t)\|_{L_4[0,T]}^2. \quad (23)$$

$$\|u(\tilde{s}_1^n(t), t)\|_{L_4[0,T]} \leq \tilde{C} \|u\|_{V_2(D)}, \quad (24)$$

# Proof of Weak Compactness in $V_2^{1,0}$

$$\left| \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx \right| \leq C_* \tilde{C} \|u^\tau\|_{V_2(D)}^2. \quad (25)$$

# Proof of Weak Compactness in $V_2^{1,0}$

$$\left| \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx \right| \leq C_* \tilde{C} \|u^\tau\|_{V_2(D)}^2. \quad (25)$$

$$C_* < (C\tilde{C})^{-1} \quad (26)$$

# Proof of Weak Compactness in $V_2^{1,0}$

$$\left| \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx \right| \leq C_* \tilde{C} \|u^\tau\|_{V_2(D)}^2. \quad (25)$$

$$C_* < (C\tilde{C})^{-1} \quad (26)$$

$$\begin{aligned} \|u^\tau\|_{V_2^{1,0}(D)}^2 \leq C & \left( \|\phi\|_{L_2(0,s_0)}^2 + \|g^n\|_{L_2(0,T)}^2 + \|f\|_{L_2(D)}^2 + \right. \\ & \left. \|\gamma(s^n(t), t)(s^n)'(t)\|_{L_2(0,T)}^2 + \|\chi(s^n(t), t)\|_{L_2(0,T)}^2 \right), \end{aligned} \quad (27)$$

# Proof of Weak Compactness in $V_2^{1,0}$

$$\left| \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx \right| \leq C_* \tilde{C} \|u^\tau\|_{V_2(D)}^2. \quad (25)$$

$$C_* < (C\tilde{C})^{-1} \quad (26)$$

$$\begin{aligned} \|u^\tau\|_{V_2^{1,0}(D)}^2 &\leq C \left( \|\phi\|_{L_2(0,s_0)}^2 + \|g^n\|_{L_2(0,T)}^2 + \|f\|_{L_2(D)}^2 + \right. \\ &\quad \left. \|\gamma(s^n(t), t)(s^n)'(t)\|_{L_2(0,T)}^2 + \|\chi(s^n(t), t)\|_{L_2(0,T)}^2 \right), \end{aligned} \quad (27)$$

$$\begin{aligned} \|\gamma(s^n(t), t)(s^n)'(t)\|_{L_2(0,T)} &\leq \|(s^n)'\|_{C[0,T]} \|\gamma(s^n(t), t)\|_{L_2[0,T]} \leq C_3 \|\gamma\|_{W_2^{1,0}(D)} \\ \|\chi(s^n(t), t)\|_{L_2[0,T]} &\leq C_3 \|\chi\|_{W_2^{1,0}(D)}, \end{aligned} \quad (28)$$

$$\|u^\tau\|_{V_2^{1,0}(D)}^2 \leq C \left( \|\phi\|_{L_2(0,s_0)}^2 + \|g\|_{L_2(0,T)}^2 + \|f\|_{L_2(D)}^2 + \|\gamma\|_{W_2^{1,0}(D)}^2 + \|\chi\|_{W_2^{1,0}(D)}^2 \right) \quad (29)$$

# Proof of Weak Compactness in $V_2^{1,0}$

The sequence  $\{u^\tau\}$  is weakly precompact in  $W_2^{1,0}(D)$ .

# Proof of Weak Compactness in $V_2^{1,0}$

The sequence  $\{u^\tau\}$  is weakly precompact in  $W_2^{1,0}(D)$ . Let  $u \in W_2^{1,0}(D)$  be a weak limit point of  $u^\tau$  in  $W_2^{1,0}(D)$ , and assume that whole sequence  $\{u^\tau\}$  converges to  $u$  weakly in  $W_2^{1,0}(D)$ .

# Proof of Weak Compactness in $V_2^{1,0}$

The sequence  $\{u^\tau\}$  is weakly precompact in  $W_2^{1,0}(D)$ . Let  $u \in W_2^{1,0}(D)$  be a weak limit point of  $u^\tau$  in  $W_2^{1,0}(D)$ , and assume that whole sequence  $\{u^\tau\}$  converges to  $u$  weakly in  $W_2^{1,0}(D)$ . Let us prove that in fact  $u$  satisfies the integral identity (7) for arbitrary test function  $\Phi \in W_2^{1,1}(\Omega)$  such that  $\Phi|_{t=T} = 0$ .

# Proof of Weak Compactness in $V_2^{1,0}$

The sequence  $\{u^\tau\}$  is weakly precompact in  $W_2^{1,0}(D)$ . Let  $u \in W_2^{1,0}(D)$  be a weak limit point of  $u^\tau$  in  $W_2^{1,0}(D)$ , and assume that whole sequence  $\{u^\tau\}$  converges to  $u$  weakly in  $W_2^{1,0}(D)$ . Let us prove that in fact  $u$  satisfies the integral identity (7) for arbitrary test function  $\Phi \in W_2^{1,1}(\Omega)$  such that  $\Phi|_{t=T} = 0$ . Due to density of  $C^1(\bar{\Omega})$  in  $W_2^{1,1}(\Omega)$  it is enough to assume  $\Phi \in C^1(\bar{\Omega})$ .  $\Phi \in C^1(\bar{D}_{T+\tau})$ ,  $\Phi \equiv 0$ , for  $T \leq t \leq T + \tau$ , where

$$D_{T+\tau} = \{(x, t) : 0 < x < l, 0 < t \leq T + \tau\}$$

$$\Phi(x; k) = \Phi(x, k\tau), \quad \Phi_t(x; k) = \frac{\Phi(x; k+1) - \Phi(x; k)}{\tau}$$

# Proof of Weak Compactness in $V_2^{1,0}$

The sequence  $\{u^\tau\}$  is weakly precompact in  $W_2^{1,0}(D)$ . Let  $u \in W_2^{1,0}(D)$  be a weak limit point of  $u^\tau$  in  $W_2^{1,0}(D)$ , and assume that whole sequence  $\{u^\tau\}$  converges to  $u$  weakly in  $W_2^{1,0}(D)$ . Let us prove that in fact  $u$  satisfies the integral identity (7) for arbitrary test function  $\Phi \in W_2^{1,1}(\Omega)$  such that  $\Phi|_{t=T} = 0$ . Due to density of  $C^1(\bar{\Omega})$  in  $W_2^{1,1}(\Omega)$  it is enough to assume  $\Phi \in C^1(\bar{\Omega})$ .  $\Phi \in C^1(\bar{D}_{T+\tau})$ ,  $\Phi \equiv 0$ , for  $T \leq t \leq T + \tau$ , where

$$D_{T+\tau} = \{(x, t) : 0 < x < l, 0 < t \leq T + \tau\}$$

$$\Phi(x; k) = \Phi(x, k\tau), \quad \Phi_t(x; k) = \frac{\Phi(x; k+1) - \Phi(x; k)}{\tau}$$

Let  $\Phi^\tau, \Phi_t^\tau$  be piecewise constant interpolations. Sequences  $\{\Phi^\tau\}$ ,  $\left\{\frac{\partial \Phi^\tau}{\partial x}\right\}$  and  $\{\Phi_t^\tau\}$  converge uniformly in  $\bar{D}$  to  $\Phi, \frac{\partial \Phi}{\partial x}$  and  $\Phi_t$  respectively.

# Proof of Weak Compactness in $V_2^{1,0}$

Choose  $\eta(x) = \tau\Phi(x; k)$  in (8), after summation with respect to  $k = \overline{1, n}$  and transformation of the time difference term as follows

# Proof of Weak Compactness in $V_2^{1,0}$

Choose  $\eta(x) = \tau\Phi(x; k)$  in (8), after summation with respect to  $k = \overline{1, n}$  and transformation of the time difference term as follows

$$\begin{aligned}
 \tau \sum_{k=1}^n \int_0^{s_k} u_{\bar{t}}(x; k) \Phi(x; k) dx &= -\tau \sum_{k=1}^{n-1} \int_0^{s_{k+1}} u(x; k) \Phi_t(x; k) dx - \\
 &\int_0^{s_1} \phi(x) \Phi(x; 1) dx - \sum_{k=1}^{n-1} \int_{s_k}^{s_{k+1}} u(x; k) \Phi(x; k) dx = \\
 &- \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_0^{s_{k+1}} u^\tau \Phi_t^\tau dx dt - \int_0^{s_1} \phi(x) \Phi^\tau(x, \tau) dx \\
 &- \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} (\tilde{s}^n)'(t) u^\tau(\tilde{s}^n(t), t - \tau) \Phi^\tau(\tilde{s}^n(t), t - \tau) dt = \\
 &- \int_0^T \int_0^{s(t)} u^\tau \Phi_t^\tau dx dt - \int_0^{s_1} \phi(x) \Phi^\tau(x, \tau) dx - \\
 &\int_0^{T-\tau} (\tilde{s}_1^n)'(t) u^\tau((\tilde{s}_1^n)(t), t) \Phi^\tau((\tilde{s}_1^n)(t), t) dt - \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \int_{s(t)}^{s_{k+1}} u^\tau \Phi_t^\tau dx dt
 \end{aligned}$$

# Proof of Weak Compactness in $V_2^{1,0}$

$$\begin{aligned}
 & \int_0^T \int_0^{s(t)} \left\{ a \frac{\partial u^\tau}{\partial x} \frac{\partial \Phi^\tau}{\partial x} - b \frac{\partial u^\tau}{\partial x} \Phi^\tau - cu^\tau \Phi^\tau + f \Phi^\tau - u^\tau \Phi_t^\tau \right\} dx dt - \\
 & \int_0^{s_0} \phi(x) \Phi^\tau(x, \tau) dx - \int_0^{T-\tau} (\tilde{s}_1^n)'(t) u^\tau((\tilde{s}_1^n)(t), t) \Phi^\tau((\tilde{s}_1^n)(t), t) dt + \\
 & \int_0^T g^n \Phi^\tau(0, t) dt + \int_0^T \left[ \gamma(s^n(t), t) (s^n)'(t) - \chi(s^n(t), t) \right] \Phi^\tau(s^n(t), t) dt = R
 \end{aligned} \tag{31}$$

,

# Proof of Weak Compactness in $V_2^{1,0}$

$$\begin{aligned}
 & \int_0^T \int_0^{s(t)} \left\{ a \frac{\partial u^\tau}{\partial x} \frac{\partial \Phi^\tau}{\partial x} - b \frac{\partial u^\tau}{\partial x} \Phi^\tau - cu^\tau \Phi^\tau + f \Phi^\tau - u^\tau \Phi_t^\tau \right\} dx dt - \\
 & \int_0^{s_0} \phi(x) \Phi^\tau(x, \tau) dx - \int_0^{T-\tau} (\tilde{s}_1^n)'(t) u^\tau((\tilde{s}_1^n)(t), t) \Phi^\tau((\tilde{s}_1^n)(t), t) dt + \\
 & \int_0^T g^n \Phi^\tau(0, t) dt + \int_0^T \left[ \gamma(s^n(t), t) (s^n)'(t) - \chi(s^n(t), t) \right] \Phi^\tau(s^n(t), t) dt = R
 \end{aligned} \tag{31}$$

$$\begin{aligned}
 R = & \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{s(t)}^{s_k} \left\{ a \frac{\partial u^\tau}{\partial x} \frac{\partial \Phi^\tau}{\partial x} - b \frac{\partial u^\tau}{\partial x} \Phi^\tau - cu^\tau \Phi^\tau + f \Phi^\tau \right\} dx dt - \\
 & \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \int_{s(t)}^{s_{k+1}} u^\tau \Phi_t^\tau dx dt + \int_{s_0}^{s_1} \phi(x) \Phi^\tau(x, \tau) dx \\
 & + \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{s^n(t)}^{s_k} \left[ \gamma(s^n(t), t) (s^n)'(t) - \chi(s^n(t), t) \right] \frac{\partial \Phi^\tau}{\partial x} dx dt
 \end{aligned}$$

# Proof of Weak Compactness in $V_2^{1,0}$

$$\Delta = \bigcup_{k=1}^n \{(x, t) : t_{k-1} < t < t_k, \min(s(t), s_k) < x < \max(s(t), s_k)\}$$

# Proof of Weak Compactness in $V_2^{1,0}$

$$\Delta = \bigcup_{k=1}^n \{(x, t) : t_{k-1} < t < t_k, \min(s(t), s_k) < x < \max(s(t), s_k)\}$$

$$|\Delta| \leq \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_t^{t_k} |s'(\tau)| d\tau dt \leq \frac{2\sqrt{T}}{3} \|s'\|_{L^2(0,T)} \tau \rightarrow 0 \quad \text{as } \tau \rightarrow 0$$

# Proof of Weak Compactness in $V_2^{1,0}$

$$\begin{aligned}\Delta &= \bigcup_{k=1}^n \{(x, t) : t_{k-1} < t < t_k, \min(s(t), s_k) < x < \max(s(t), s_k)\} \\ |\Delta| &\leq \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_t^{t_k} |s'(\tau)| d\tau dt \leq \frac{2\sqrt{T}}{3} \|s'\|_{L^2(0,T)} \tau \rightarrow 0 \quad \text{as } \tau \rightarrow 0 \\ \lim_{\tau \rightarrow 0} R &= 0\end{aligned}\tag{32}$$

# Proof of Weak Compactness in $V_2^{1,0}$

$$\Delta = \bigcup_{k=1}^n \{(x, t) : t_{k-1} < t < t_k, \min(s(t), s_k) < x < \max(s(t), s_k)\}$$

$$|\Delta| \leq \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_t^{t_k} |s'(\tau)| d\tau dt \leq \frac{2\sqrt{T}}{3} \|s'\|_{L^2(0,T)} \tau \rightarrow 0 \quad \text{as } \tau \rightarrow 0$$

$$\lim_{\tau \rightarrow 0} R = 0 \quad (32)$$

$$\|u^\tau(\tilde{s}_1^n(t), t) - u^\tau(s(t), t)\|_{L_2[0,T]} \rightarrow 0 \quad \text{as } \tau \rightarrow 0 \quad (33)$$

$$u^\tau(s(t), t) \rightarrow u(s(t), t), \quad \text{weakly in } L_2[0, T] \quad (34)$$

$$\lim_{\tau \rightarrow 0} \int_0^{T-\tau} (\tilde{s}_1^n)'(t) u^\tau(\tilde{s}_1^n(t), t) \Phi^\tau(\tilde{s}_1^n(t), t) dt = \int_0^T s'(t) u(s(t), t) \Phi(s(t), t) dt \quad (35)$$

# Proof of Weak Compactness in $V_2^{1,0}$

$$\Delta = \bigcup_{k=1}^n \{(x, t) : t_{k-1} < t < t_k, \min(s(t), s_k) < x < \max(s(t), s_k)\}$$

$$|\Delta| \leq \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_t^{t_k} |s'(\tau)| d\tau dt \leq \frac{2\sqrt{T}}{3} \|s'\|_{L^2(0,T)} \tau \rightarrow 0 \quad \text{as } \tau \rightarrow 0$$

$$\lim_{\tau \rightarrow 0} R = 0 \quad (32)$$

$$\|u^\tau(\tilde{s}_1^n(t), t) - u^\tau(s(t), t)\|_{L_2[0,T]} \rightarrow 0 \quad \text{as } \tau \rightarrow 0 \quad (33)$$

$$u^\tau(s(t), t) \rightarrow u(s(t), t), \quad \text{weakly in } L_2[0, T] \quad (34)$$

$$\lim_{\tau \rightarrow 0} \int_0^{T-\tau} (\tilde{s}_1^n)'(t) u^\tau(\tilde{s}_1^n(t), t) \Phi^\tau(\tilde{s}_1^n(t), t) dt = \int_0^T s'(t) u(s(t), t) \Phi(s(t), t) dt \quad (35)$$

Passing to the limit as  $\tau \rightarrow 0$ , from (21) it follows that  $u$  is a weak solution of the problem (2)-(5). Whole sequence  $\{u^\tau\}$  converges to  $u \in V_2^{1,0}(\Omega)$  weakly in  $W_2^{1,0}(\Omega)$ . Theorem is proved.

# Proof of Weak Compactness in $V_2^{1,0}$

$$\Delta = \bigcup_{k=1}^n \{(x, t) : t_{k-1} < t < t_k, \min(s(t), s_k) < x < \max(s(t), s_k)\}$$

$$|\Delta| \leq \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_t^{t_k} |s'(\tau)| d\tau dt \leq \frac{2\sqrt{T}}{3} \|s'\|_{L^2(0,T)} \tau \rightarrow 0 \quad \text{as } \tau \rightarrow 0$$

$$\lim_{\tau \rightarrow 0} R = 0 \tag{32}$$

$$\|u^\tau(\tilde{s}_1^n(t), t) - u^\tau(s(t), t)\|_{L_2[0,T]} \rightarrow 0 \quad \text{as } \tau \rightarrow 0 \tag{33}$$

$$u^\tau(s(t), t) \rightarrow u(s(t), t), \quad \text{weakly in } L_2[0, T] \tag{34}$$

$$\lim_{\tau \rightarrow 0} \int_0^{T-\tau} (\tilde{s}_1^n)'(t) u^\tau(\tilde{s}_1^n(t), t) \Phi^\tau(\tilde{s}_1^n(t), t) dt = \int_0^T s'(t) u(s(t), t) \Phi(s(t), t) dt \tag{35}$$

Passing to the limit as  $\tau \rightarrow 0$ , from (21) it follows that  $u$  is a weak solution of the problem (2)-(5). Whole sequence  $\{u^\tau\}$  converges to  $u \in V_2^{1,0}(\Omega)$  weakly in  $W_2^{1,0}(\Omega)$ . Theorem is proved.

## Corollary 12

*For arbitrary  $v = (s, g) \in V_R$  there exists a weak solution  $u \in V_2^{1,0}(\Omega)$  of the problem (2)-(5) which satisfy the energy estimate (12):*

$$\|u\|_{V_2^{1,0}(D)}^2 \leq C \left( \|\phi\|_{L_2(0,s_0)}^2 + \|g\|_{L_2(0,T)}^2 + \|f\|_{L_2(D)}^2 + \|\gamma\|_{W_2^{1,0}(D)}^2 + \|\chi\|_{W_2^{1,0}(D)}^2 \right)$$

## Corollary 12

For arbitrary  $v = (s, g) \in V_R$  there exists a weak solution  $u \in V_2^{1,0}(\Omega)$  of the problem (2)-(5) which satisfy the energy estimate (12):

$$\|u\|_{V_2^{1,0}(D)}^2 \leq C \left( \|\phi\|_{L_2(0,s_0)}^2 + \|g\|_{L_2(0,T)}^2 + \|f\|_{L_2(D)}^2 + \|\gamma\|_{W_2^{1,0}(D)}^2 + \|\chi\|_{W_2^{1,0}(D)}^2 \right)$$

Assumptions needed for the weak compactness in  $V_2^{1,0}(\Omega)$ :

$$\phi \in L_2[0, l], \quad \gamma, \chi \in W_2^{1,0}(D), \quad a \in L_\infty(D)$$

## Theorem 13

*For all sufficiently small  $\tau$  discrete state vector  $[u([v]_n)]_n$  satisfies the following stability estimation:*

## Theorem 13

For all sufficiently small  $\tau$  discrete state vector  $[u([v]_n)]_n$  satisfies the following stability estimation:

$$\begin{aligned} \max_{1 \leq k \leq n} \int_0^l \left| \frac{du(x; k)}{dx} \right|^2 dx + \tau \sum_{k=1}^n \int_0^l u_t^2(x; k) dx \leq C \left[ \|\phi\|_{W_2^1[0, l]}^2 + \right. \\ \left. \|g^n\|_{W_2^{\frac{1}{4}}[0, T]}^2 + \|\gamma(s^n(t), t)(s^n)'(t)\|_{W_2^{\frac{1}{4}}[0, T]}^2 + \|\chi(s^n(t), t)\|_{W_2^{\frac{1}{4}}[0, T]}^2 + \right. \\ \left. + \|f\|_{L_2(D)}^2 + \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx \right] \quad (36) \end{aligned}$$

# Second Energy Estimate and its Consequences

## Theorem 13

For all sufficiently small  $\tau$  discrete state vector  $[u([v]_n)]_n$  satisfies the following stability estimation:

$$\begin{aligned} \max_{1 \leq k \leq n} \int_0^l \left| \frac{du(x; k)}{dx} \right|^2 dx + \tau \sum_{k=1}^n \int_0^l u_t^2(x; k) dx \leq C \left[ \|\phi\|_{W_2^1[0, l]}^2 + \right. \\ \left. \|g^n\|_{W_2^{\frac{1}{4}}[0, T]}^2 + \|\gamma(s^n(t), t)(s^n)'(t)\|_{W_2^{\frac{1}{4}}[0, T]}^2 + \|\chi(s^n(t), t)\|_{W_2^{\frac{1}{4}}[0, T]}^2 + \right. \\ \left. + \|f\|_{L_2(D)}^2 + \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx \right] \quad (36) \end{aligned}$$

$W_2^{\frac{1}{4}}[0, T]$  – Banach space of all elements of  $L_2[0, T]$  with finite norm

$$\|u\|_{W_2^{\frac{1}{4}}[0, T]} = \left( \|u\|_{L_2[0, T]}^2 + \int_0^T dt \int_0^T \frac{|u(t) - u(\tau)|^2}{|t - \tau|^{\frac{3}{2}}} d\tau \right)^{\frac{1}{2}}$$

## Lemma 14

Given  $[v]_n$  define the vector function

$$[\tilde{u}([v]_n)]_n = (\tilde{u}(x; 0), \tilde{u}(x; 1), \dots, \tilde{u}(x; n))$$

$$\tilde{u}(x; k) = \begin{cases} u(x; k) & 0 \leq x \leq s_k, \\ u(s_k; k) & s_k \leq x \leq l, k = \overline{0, n}. \end{cases}$$

## Lemma 14

Given  $[v]_n$  define the vector function

$$[\tilde{u}([v]_n)]_n = (\tilde{u}(x; 0), \tilde{u}(x; 1), \dots, \tilde{u}(x; n))$$

$$\tilde{u}(x; k) = \begin{cases} u(x; k) & 0 \leq x \leq s_k, \\ u(s_k; k) & s_k \leq x \leq l, k = \overline{0, n}. \end{cases}$$

Then for all sufficiently small  $\tau$ ,  $[\tilde{u}([v]_n)]_n$  satisfies the following estimation:

## Lemma 14

Given  $[v]_n$  define the vector function

$$[\tilde{u}([v]_n)]_n = (\tilde{u}(x; 0), \tilde{u}(x; 1), \dots, \tilde{u}(x; n))$$

$$\tilde{u}(x; k) = \begin{cases} u(x; k) & 0 \leq x \leq s_k, \\ u(s_k; k) & s_k \leq x \leq l, k = \overline{0, n}. \end{cases}$$

Then for all sufficiently small  $\tau$ ,  $[\tilde{u}([v]_n)]_n$  satisfies the following estimation:

$$\begin{aligned} & \max_{1 \leq k \leq n} \int_0^{s_k} \left| \frac{d\tilde{u}(x; k)}{dx} \right|^2 dx + \tau \sum_{k=1}^m \int_0^{s_k} \tilde{u}_{\bar{t}}^2(x; k) dx + \\ & \tau^2 \sum_{k=1}^m \int_0^{s_k} \left[ \left( \frac{d\tilde{u}(x; k)}{dx} \right)_{\bar{t}} \right]^2 dx \leq C \left[ \|\phi\|_{W_2^1[0, l]}^2 + \|g^n\|_{W_2^{\frac{1}{4}}[0, T]}^2 \right. \\ & \left. + \|\gamma(s^n(t), t)(s^n)'(t)\|_{W_2^{\frac{1}{4}}[0, T]}^2 + \|\chi(s^n(t), t)\|_{W_2^{\frac{1}{4}}[0, T]}^2 + \|f\|_{L_2(D)}^2 \right. \\ & \left. + \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx \right], \quad (37) \end{aligned}$$

**Proof:** Choose  $\eta(x) = 2\tau\tilde{u}_{\bar{t}}(x; k)$  in (8) and use the identity

$$2\tau a_k(x) \frac{d\tilde{u}(x; k)}{dx} \left( \frac{d\tilde{u}(x; k)}{dx} \right)_{\bar{t}} = a_k(x) \left( \frac{d\tilde{u}(x; k)}{dx} \right)^2 - a_{k-1}(x) \left( \frac{d\tilde{u}(x; k-1)}{dx} \right) - \tau a_{k\bar{t}}(x) \left( \frac{d\tilde{u}(x; k-1)}{dx} \right)^2 + \tau^2 a_k(x) \left[ \left( \frac{d\tilde{u}(x; k)}{dx} \right)_{\bar{t}} \right]^2, \quad (38)$$

**Proof:** Choose  $\eta(x) = 2\tau\tilde{u}_{\bar{t}}(x; k)$  in (8) and use the identity

$$2\tau a_k(x) \frac{d\tilde{u}(x; k)}{dx} \left( \frac{d\tilde{u}(x; k)}{dx} \right)_{\bar{t}} = a_k(x) \left( \frac{d\tilde{u}(x; k)}{dx} \right)^2 - a_{k-1}(x) \left( \frac{d\tilde{u}(x; k-1)}{dx} \right)^2 - \tau a_{k\bar{t}}(x) \left( \frac{d\tilde{u}(x; k-1)}{dx} \right)^2 + \tau^2 a_k(x) \left[ \left( \frac{d\tilde{u}(x; k)}{dx} \right)_{\bar{t}} \right]^2, \quad (38)$$

$$\begin{aligned} & \int_0^{s_k} a_k(x) \left( \frac{d\tilde{u}(x; k)}{dx} \right)^2 dx - \int_0^{s_{k-1}} a_{k-1}(x) \left( \frac{d\tilde{u}(x; k-1)}{dx} \right)^2 dx + \\ & 2\tau \int_0^{s_k} (\tilde{u}_{\bar{t}}(x; k))^2 dx + \tau^2 \int_0^{s_k} a_k(x) \left[ \left( \frac{d\tilde{u}(x; k)}{dx} \right)_{\bar{t}} \right]^2 \leq \\ & \tau \int_0^{s_k} a_{k\bar{t}}(x) \left( \frac{d\tilde{u}(x; k-1)}{dx} \right)^2 dx + 2\tau \int_0^{s_k} b_k(x) \frac{d\tilde{u}(x; k)}{dx} \tilde{u}_{\bar{t}}(x; k) dx \\ & + 2\tau \int_0^{s_k} c_k(x) \tilde{u}(x; k) \tilde{u}_{\bar{t}}(x; k) dx - 2\tau \int_0^{s_k} f_k(x) u_{\bar{t}}(x; k) dx \\ & - 2\tau [(\gamma_{s^n}(s^n)')^k - \chi_{s^n}^k] \tilde{u}_{\bar{t}}(s_k; k) - 2\tau g_k^n \tilde{u}_{\bar{t}}(0; k) \end{aligned} \quad (39)$$

By adding inequalities (39) with respect to  $k$  from 1 to arbitrary  $m \leq n$  we derive

By adding inequalities (39) with respect to  $k$  from 1 to arbitrary  $m \leq n$  we derive

$$\begin{aligned}
 & \int_0^{s_m} a_m(x) \left( \frac{d\tilde{u}(x; m)}{dx} \right)^2 dx + 2\tau \sum_{k=1}^m \int_0^{s_k} \tilde{u}_{\bar{t}}^2(x; k) dx + \\
 & \tau^2 \sum_{k=1}^m \int_0^{s_k} a_k(x) \left[ \left( \frac{d\tilde{u}(x; k)}{dx} \right)_{\bar{t}} \right]^2 dx \leq \tau \sum_{k=1}^m \int_0^{s_k} a_{k\bar{t}}(x) \left( \frac{d\tilde{u}(x; k-1)}{dx} \right)^2 dx \\
 & + 2\tau \sum_{k=1}^m \int_0^{s_k} b_k(x) \frac{d\tilde{u}(x; k)}{dx} \tilde{u}_{\bar{t}}(x; k) dx + 2\tau \sum_{k=1}^m \int_0^{s_k} c_k(x) \tilde{u}(x; k) \tilde{u}_{\bar{t}}(x; k) dx \\
 & - 2\tau \sum_{k=1}^m \int_0^{s_k} f_k(x) \tilde{u}_{\bar{t}}(x; k) dx + \int_0^{s_0} a_0(x) \left( \frac{d\phi}{dx} \right)^2 dx - \\
 & 2\tau \sum_{k=1}^m [(\gamma_{s^n}(s^n)')^k - \chi_{s^n}^k] \tilde{u}_{\bar{t}}(s_k; k) - 2\tau \sum_{k=1}^m g_k^n \tilde{u}_{\bar{t}}(0; k) \quad (40)
 \end{aligned}$$

$$\begin{aligned}
& a_0 \int_0^{s_m} \left| \frac{d\tilde{u}(x; k)}{dx} \right|^2 dx + \tau \sum_{k=1}^m \int_0^{s_k} \tilde{u}_{\bar{t}}^2(x; k) dx + \\
& a_0 \tau^2 \sum_{k=1}^m \int_0^{s_k} \left[ \left( \frac{d\tilde{u}(x; k)}{dx} \right)_{\bar{t}} \right]^2 dx \leq C \tau \sum_{k=1}^m \left[ \int_0^{s_k} u^2(x; k) dx + \right. \\
& \left. \int_0^{s_k} \left| \frac{du(x; k)}{dx} \right|^2 dx + \int_0^{s_k} f_k^2(x) dx \right] C \int_0^{s_0} \left| \frac{d\phi}{dx} \right|^2 dx - \\
& 2\tau \sum_{k=1}^m [(\gamma_{s^n}(s^n)')^k - \chi_{s^n}^k] \tilde{u}_{\bar{t}}(s_k; k) - 2\tau \sum_{k=1}^m g_k^n \tilde{u}_{\bar{t}}(0; k) dx \quad (41)
\end{aligned}$$

$$\begin{aligned}
& a_0 \int_0^{s_m} \left| \frac{d\tilde{u}(x; k)}{dx} \right|^2 dx + \tau \sum_{k=1}^m \int_0^{s_k} \tilde{u}_{\bar{t}}^2(x; k) dx + \\
& a_0 \tau^2 \sum_{k=1}^m \int_0^{s_k} \left[ \left( \frac{d\tilde{u}(x; k)}{dx} \right)_{\bar{t}} \right]^2 dx \leq C \tau \sum_{k=1}^m \left[ \int_0^{s_k} u^2(x; k) dx + \right. \\
& \left. \int_0^{s_k} \left| \frac{du(x; k)}{dx} \right|^2 dx + \int_0^{s_k} f_k^2(x) dx \right] C \int_0^{s_0} \left| \frac{d\phi}{dx} \right|^2 dx - \\
& 2\tau \sum_{k=1}^m [(\gamma_{s^n}(s^n))' ]^k - \chi_{s^n}^k] \tilde{u}_{\bar{t}}(s_k; k) - 2\tau \sum_{k=1}^m g_k^n \tilde{u}_{\bar{t}}(0; k) dx \quad (41)
\end{aligned}$$

$$\|\gamma(s^n(t), t)\|_{W_2^{\frac{1}{4}}[0, T]} \leq C \|\gamma\|_{W_2^{1,1}(D)}, \quad \|\chi(s^n(t), t)\|_{W_2^{\frac{1}{4}}[0, T]} \leq C \|\chi\|_{W_2^{1,1}(D)}, \quad (42)$$

$$\begin{aligned}
\|\gamma(s^n(t), t)(s^n)'(t)\|_{W_2^{\frac{1}{4}}[0, T]} & \leq C_1 \|\gamma(s^n(t), t)\|_{W_2^{\frac{1}{4}}[0, T]} \|s^n\|_{W_2^2[0, T]} \\
& \leq C \|\gamma\|_{W_2^{1,1}(D)}, \quad (43)
\end{aligned}$$

Let  $w(x, t)$  be a function in  $W_2^{2,1}(D)$  such that

$$w(x, 0) = \phi(x) \quad \text{for } x \in [0, s_0], \quad a(0, t)w_x(0, t) = g^n(t), \quad \text{for a.e. } t \in [0, T] \quad (44)$$

$$a(s^n(t), t)w_x(s^n(t), t) = \gamma(s^n(t), t)(s^n)'(t) - \chi(s^n(t), t) \quad \text{for a.e. } t \in [0, T] \quad (45)$$

Let  $w(x, t)$  be a function in  $W_2^{2,1}(D)$  such that

$$w(x, 0) = \phi(x) \quad \text{for } x \in [0, s_0], \quad a(0, t)w_x(0, t) = g^n(t), \quad \text{for a.e. } t \in [0, T] \quad (44)$$

$$a(s^n(t), t)w_x(s^n(t), t) = \gamma(s^n(t), t)(s^n)'(t) - \chi(s^n(t), t) \quad \text{for a.e. } t \in [0, T] \quad (45)$$

$$\begin{aligned} \|w\|_{W_2^{2,1}(D)} \leq C & \left[ \|g^n\|_{W_2^{\frac{1}{4}}[0, T]} + \|\phi(x)\|_{W_2^1[0, s_0]} \right. \\ & \left. + \|\gamma(s^n(t), t)(s^n)'(t) - \chi(s^n(t), t)\|_{W_2^{\frac{1}{4}}[0, T]} \right] \quad (46) \end{aligned}$$

Let  $w(x, t)$  be a function in  $W_2^{2,1}(D)$  such that

$$w(x, 0) = \phi(x) \quad \text{for } x \in [0, s_0], \quad a(0, t)w_x(0, t) = g^n(t), \quad \text{for a.e. } t \in [0, T] \quad (44)$$

$$a(s^n(t), t)w_x(s^n(t), t) = \gamma(s^n(t), t)(s^n)'(t) - \chi(s^n(t), t) \quad \text{for a.e. } t \in [0, T] \quad (45)$$

$$\begin{aligned} \|w\|_{W_2^{2,1}(D)} \leq C & \left[ \|g^n\|_{W_2^{\frac{1}{4}}[0, T]} + \|\phi(x)\|_{W_2^1[0, s_0]} \right. \\ & \left. + \|\gamma(s^n(t), t)(s^n)'(t) - \chi(s^n(t), t)\|_{W_2^{\frac{1}{4}}[0, T]} \right] \quad (46) \end{aligned}$$

$w$  can be constructed as a solution from  $W_2^{2,1}(\Omega^n)$  of the heat eq. in

$$\Omega^n = \{0 < x < s^n(t), 0 < t < T\}.$$

Let  $w(x, t)$  be a function in  $W_2^{2,1}(D)$  such that

$$w(x, 0) = \phi(x) \quad \text{for } x \in [0, s_0], \quad a(0, t)w_x(0, t) = g^n(t), \quad \text{for a.e. } t \in [0, T] \quad (44)$$

$$a(s^n(t), t)w_x(s^n(t), t) = \gamma(s^n(t), t)(s^n)'(t) - \chi(s^n(t), t) \quad \text{for a.e. } t \in [0, T] \quad (45)$$

$$\begin{aligned} \|w\|_{W_2^{2,1}(D)} \leq C & \left[ \|g^n\|_{W_2^{\frac{1}{4}}[0, T]} + \|\phi(x)\|_{W_2^1[0, s_0]} \right. \\ & \left. + \|\gamma(s^n(t), t)(s^n)'(t) - \chi(s^n(t), t)\|_{W_2^{\frac{1}{4}}[0, T]} \right] \quad (46) \end{aligned}$$

$w$  can be constructed as a solution from  $W_2^{2,1}(\Omega^n)$  of the heat eq. in

$$\Omega^n = \{0 < x < s^n(t), 0 < t < T\}.$$

By replacing in original Neumann problem  $s \rightarrow s^n, g \rightarrow g^n, u \rightarrow u - w$  derive (41) without last three terms and with  $f_k(x)$  replaced by Steklov average  $F_k(x)$  of

$$F = f + w_t - (aw_x)_x - bw_x - cw \in L_2(D). \quad (47)$$

Let  $w(x, t)$  be a function in  $W_2^{2,1}(D)$  such that

$$w(x, 0) = \phi(x) \quad \text{for } x \in [0, s_0], \quad a(0, t)w_x(0, t) = g^n(t), \quad \text{for a.e. } t \in [0, T] \quad (44)$$

$$a(s^n(t), t)w_x(s^n(t), t) = \gamma(s^n(t), t)(s^n)'(t) - \chi(s^n(t), t) \quad \text{for a.e. } t \in [0, T] \quad (45)$$

$$\begin{aligned} \|w\|_{W_2^{2,1}(D)} \leq C & \left[ \|g^n\|_{W_2^{\frac{1}{4}}[0, T]} + \|\phi(x)\|_{W_2^1[0, s_0]} \right. \\ & \left. + \|\gamma(s^n(t), t)(s^n)'(t) - \chi(s^n(t), t)\|_{W_2^{\frac{1}{4}}[0, T]} \right] \quad (46) \end{aligned}$$

$w$  can be constructed as a solution from  $W_2^{2,1}(\Omega^n)$  of the heat eq. in

$$\Omega^n = \{0 < x < s^n(t), 0 < t < T\}.$$

By replacing in original Neumann problem  $s \rightarrow s^n, g \rightarrow g^n, u \rightarrow u - w$  derive (41) without last three terms and with  $f_k(x)$  replaced by Steklov average  $F_k(x)$  of

$$F = f + w_t - (aw_x)_x - bw_x - cw \in L_2(D). \quad (47)$$

From modified (41),(46),(47), desired (37) follows. Lemma is proved.

In the next lemma we prove the stability estimation with  $l$  being replaced by  $s_k$ .

In the next lemma we prove the stability estimation with  $l$  being replaced by  $s_k$ .

### Lemma 15

For all sufficiently small  $\tau$ , discrete state vector  $[u([v]_n)]_n$  satisfies the following estimation:

$$\begin{aligned} & \max_{1 \leq k \leq n} \int_0^{s_k} \left| \frac{du(x; k)}{dx} \right|^2 dx + \tau \sum_{k=1}^n \int_0^{s_k} u_t^2(x; k) dx \leq \\ & C \left[ \|\phi\|_{W_2^1[0, l]}^2 + \|g^n\|_{W_2^{\frac{1}{4}}[0, T]}^2 + \|\gamma(s^n(t), t)(s^n)'(t)\|_{W_2^{\frac{1}{4}}[0, T]}^2 + \right. \\ & \left. \|\chi(s^n(t), t)\|_{W_2^{\frac{1}{4}}[0, T]}^2 + \|f\|_{L_2(D)}^2 + \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x; k) dx \right], \end{aligned} \quad (48)$$

we only need to estimate  $\int_0^{s_k} u_{\bar{t}}^2(x; k) dx$ ,  $s_{k-1} < s_k$ .

we only need to estimate  $\int_0^{s_k} u_{\bar{t}}^2(x; k) dx$ ,  $s_{k-1} < s_k$ .

$$\int_0^{s_k} u_{\bar{t}}^2(x; k) dx = \int_0^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx + \int_{s_{k-1}}^{s_k} u_{\bar{t}}^2(x; k) dx$$

$$\int_{s_{k-1}}^{s_k} \left| \frac{u(x; k) - u(x; k-1)}{\tau} \right|^2 dx \leq 2 \int_{s_{k-1}}^{s_k} \left| \frac{u(x; k) - u(2s_{k-1} - x; k)}{\tau} \right|^2 dx$$

$$+ 2 \int_{s_{k-1}}^{s_k} \left| \frac{u(2s_{k-1} - x; k) - u(2s_{k-1} - x; k-1)}{\tau} \right|^2 dx \leq$$

$$2 \int_{s_{k-1}}^{s_k} \left| \frac{1}{\tau} \int_{2s_{k-1}-x}^x \frac{du(y; k)}{dy} \right|^2 dx + 2 \int_{s_{k-1} - (s_k - s_{k-1})}^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx \leq$$

$$\frac{2}{\tau^2} \int_{s_{k-1}}^{s_k} \int_{2s_{k-1}-x}^x \left| \frac{du(y; k)}{dy} \right|^2 dy 2(x - s_{k-1}) dx + 2 \int_{s_{k-1} - C\tau}^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx \leq$$

we only need to estimate  $\int_0^{s_k} u_{\bar{t}}^2(x; k) dx$ ,  $s_{k-1} < s_k$ .

$$\int_0^{s_k} u_{\bar{t}}^2(x; k) dx = \int_0^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx + \int_{s_{k-1}}^{s_k} u_{\bar{t}}^2(x; k) dx$$

$$\int_{s_{k-1}}^{s_k} \left| \frac{u(x; k) - u(x; k-1)}{\tau} \right|^2 dx \leq 2 \int_{s_{k-1}}^{s_k} \left| \frac{u(x; k) - u(2s_{k-1} - x; k)}{\tau} \right|^2 dx$$

$$+ 2 \int_{s_{k-1}}^{s_k} \left| \frac{u(2s_{k-1} - x; k) - u(2s_{k-1} - x; k-1)}{\tau} \right|^2 dx \leq$$

$$2 \int_{s_{k-1}}^{s_k} \left| \frac{1}{\tau} \int_{2s_{k-1}-x}^x \frac{du(y; k)}{dy} \right|^2 dx + 2 \int_{s_{k-1} - (s_k - s_{k-1})}^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx \leq$$

$$\frac{2}{\tau^2} \int_{s_{k-1}}^{s_k} \int_{2s_{k-1}-x}^x \left| \frac{du(y; k)}{dy} \right|^2 dy 2(x - s_{k-1}) dx + 2 \int_{s_{k-1} - C\tau}^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx \leq$$

$$2 \int_{s_{k-1}-C\tau}^{s_k} \left| \frac{d\tilde{u}(x; k)}{dx} \right|^2 dx + 2 \int_{s_{k-1}-C\tau}^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx. \quad (49)$$

$$2 \int_{s_{k-1}-C\tau}^{s_k} \left| \frac{d\tilde{u}(x; k)}{dx} \right|^2 dx + 2 \int_{s_{k-1}-C\tau}^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx. \quad (49)$$

Hence, for sufficiently small  $\tau$  we have

$$\begin{aligned} \int_0^{s_k} u_{\bar{t}}^2(x; k) dx &\leq 2 \int_{s_{k-1}-C\tau}^{s_k} \left| \frac{d\tilde{u}(x; k)}{dx} \right|^2 dx + \int_0^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx \\ + 2 \int_{s_{k-1}-C\tau}^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx &\leq 2 \int_0^{s_k} \left| \frac{d\tilde{u}(x; k)}{dx} \right|^2 dx + 3 \int_0^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx. \quad (50) \end{aligned}$$

$$2 \int_{s_{k-1}-C\tau}^{s_k} \left| \frac{d\tilde{u}(x; k)}{dx} \right|^2 dx + 2 \int_{s_{k-1}-C\tau}^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx. \quad (49)$$

Hence, for sufficiently small  $\tau$  we have

$$\begin{aligned} \int_0^{s_k} u_{\bar{t}}^2(x; k) dx &\leq 2 \int_{s_{k-1}-C\tau}^{s_k} \left| \frac{d\tilde{u}(x; k)}{dx} \right|^2 dx + \int_0^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx \\ + 2 \int_{s_{k-1}-C\tau}^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx &\leq 2 \int_0^{s_k} \left| \frac{d\tilde{u}(x; k)}{dx} \right|^2 dx + 3 \int_0^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx. \end{aligned} \quad (50)$$

Lemma is proved.

$$2 \int_{s_{k-1}-C\tau}^{s_k} \left| \frac{d\tilde{u}(x; k)}{dx} \right|^2 dx + 2 \int_{s_{k-1}-C\tau}^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx. \quad (49)$$

Hence, for sufficiently small  $\tau$  we have

$$\begin{aligned} \int_0^{s_k} u_{\bar{t}}^2(x; k) dx &\leq 2 \int_{s_{k-1}-C\tau}^{s_k} \left| \frac{d\tilde{u}(x; k)}{dx} \right|^2 dx + \int_0^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx \\ + 2 \int_{s_{k-1}-C\tau}^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx &\leq 2 \int_0^{s_k} \left| \frac{d\tilde{u}(x; k)}{dx} \right|^2 dx + 3 \int_0^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx. \quad (50) \end{aligned}$$

Lemma is proved. By using extension lemma, from Lemma 15 second energy estimate follows.

$$2 \int_{s_{k-1}-C\tau}^{s_k} \left| \frac{d\tilde{u}(x; k)}{dx} \right|^2 dx + 2 \int_{s_{k-1}-C\tau}^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx. \quad (49)$$

Hence, for sufficiently small  $\tau$  we have

$$\begin{aligned} \int_0^{s_k} u_{\bar{t}}^2(x; k) dx &\leq 2 \int_{s_{k-1}-C\tau}^{s_k} \left| \frac{d\tilde{u}(x; k)}{dx} \right|^2 dx + \int_0^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx \\ + 2 \int_{s_{k-1}-C\tau}^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx &\leq 2 \int_0^{s_k} \left| \frac{d\tilde{u}(x; k)}{dx} \right|^2 dx + 3 \int_0^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x; k) dx. \quad (50) \end{aligned}$$

Lemma is proved. By using extension lemma, from Lemma 15 second energy estimate follows. Theorem is proved.

## Theorem 16

Let  $[v]_n \in V_R^n, n = 1, 2, \dots$  be a sequence of discrete controls and the sequence  $\{\mathcal{P}_n([v]_n)\}$  converges weakly in  $W_2^2[0, T] \times W_2^1[0, T]$  to  $v = (s, g)$ . Then the sequence  $\{\hat{u}^\tau\}$  converges as  $\tau \rightarrow 0$  weakly in  $W_2^{1,1}(\Omega)$  to weak solution  $u \in W_2^{1,1}(\Omega)$  of the problem (2)-(5), i.e. to the solution of the integral identity (6). Moreover,  $u$  satisfies the energy estimate

$$\|u\|_{W_2^{1,1}(D)}^2 \leq C \left( \|\phi\|_{W_2^1(0, s_0)}^2 + \|g\|_{W_2^{\frac{1}{4}}}^2 + \|f\|_{L_2(D)}^2 + \|\gamma\|_{W_2^{1,1}(D)}^2 + \|\chi\|_{W_2^{1,1}(D)}^2 \right) \quad (51)$$

## Theorem 16

Let  $[v]_n \in V_R^n, n = 1, 2, \dots$  be a sequence of discrete controls and the sequence  $\{\mathcal{P}_n([v]_n)\}$  converges weakly in  $W_2^2[0, T] \times W_2^1[0, T]$  to  $v = (s, g)$ . Then the sequence  $\{\hat{u}^\tau\}$  converges as  $\tau \rightarrow 0$  weakly in  $W_2^{1,1}(\Omega)$  to weak solution  $u \in W_2^{1,1}(\Omega)$  of the problem (2)-(5), i.e. to the solution of the integral identity (6). Moreover,  $u$  satisfies the energy estimate

$$\|u\|_{W_2^{1,1}(D)}^2 \leq C \left( \|\phi\|_{W_2^1(0,s_0)}^2 + \|g\|_{W_2^{\frac{1}{4}}}^2 + \|f\|_{L_2(D)}^2 + \|\gamma\|_{W_2^{1,1}(D)}^2 + \|\chi\|_{W_2^{1,1}(D)}^2 \right) \quad (51)$$

## Corollary 17

For arbitrary  $v = (s, g) \in V_R$  there exists a weak solution  $u \in W_2^{1,1}(\Omega)$  of the problem (2)-(5) which satisfy the energy estimate

$$\|u\|_{W_2^{1,1}(D)}^2 \leq C \left( \|\phi\|_{W_2^1(0,s_0)}^2 + \|g\|_{W_2^{\frac{1}{4}}}^2 + \|f\|_{L_2(D)}^2 + \|\gamma\|_{W_2^{1,1}(D)}^2 + \|\chi\|_{W_2^{1,1}(D)}^2 \right)$$