Optimal Control and Inverse Problems for PDEs. Inverse Stefan Problem - Part 2

Ugur G. Abdulla

Florida Institute of Technology

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Inverse Stefan Problem (ISP): *Find the functions* $u(x,t)$ *and* $s(t)$ *and the boundary heat flux* $g(t)$ *satisfying conditions*

$$(a(x,t)u_x)_x + b(x,t)u_x + c(x,t)u - u_t = f(x,t), \quad \text{for } (x,t) \in \Omega$$

$$u(x,0) = \phi(x), \qquad 0 \le x \le s(0) = s_0$$

$$a(0,t)u_x(0,t) = g(t), \qquad 0 \le t \le T$$

$$a(s(t),t)u_x(s(t),t) + \gamma(s(t),t)s'(t) = \chi(s(t),t), \qquad 0 \le t \le T$$

$$u(s(t),t) = \mu(t), \qquad 0 \le t \le T$$

$$u(0,t) = \nu(t), \quad \text{for } 0 \le t \le T$$

Optimal Control Problem

$$\mathcal{J}(v) = \beta_0 \|u(0,t) - \nu(t)\|^2_{L_2[0,T]} + \beta_1 \|u(s(t),t) - \mu(t)\|^2_{L_2[0,T]} \qquad (1)$$

$$V_R = \Big\{ v = (s,g) \in W_2^2[0,T] \times W_2^1[0,T] : \delta \le s(t) \le l, \\ s(0) = s_0, \max(\ \|s\|_{W_2^2};\ \|g\|_{W_2^1}) \le R \Big\}$$

$$(a(x,t)u_x)_x + b(x,t)u_x + c(x,t)u - u_t = f(x,t), \quad \text{for } (x,t) \in \Omega \qquad (2)$$

$$u(x,0) = \phi(x), \qquad 0 \le x \le s(0) = s_0 \qquad (3)$$

$$a(0,t)u_x(0,t) = g(t), \qquad 0 \le t \le T \qquad (4)$$

$$a(s(t),t)u_x(s(t),t) + \gamma(s(t),t)s'(t) = \chi(s(t),t), \qquad 0 \le t \le T \qquad (5)$$

$$\Omega = \{(x,t) :\ 0 < x < s(t),\ 0 < t \le T\}$$

$W_2^{1,1}(\Omega)$-solution of the Neumann Problem

Definition 1

The function $u \in W_2^{1,1}(\Omega)$ is called a weak solution of the problem (2)-(5) if $u(x,0) = \phi(x) \in W_2^1[0, s_0]$ and

$$0 = \int_0^T \int_0^{s(t)} [au_x \Phi_x - bu_x \Phi - cu\Phi + u_t \Phi + f\Phi] \, dx \, dt$$
$$+ \int_0^T [\gamma(s(t),t)s'(t) - \chi(s(t),t)]\Phi(s(t),t) \, dt + \int_0^T g(t)\Phi(0,t) \, dt \quad (6)$$

for arbitrary $\Phi \in W_2^{1,1}(\Omega)$

$W_2^{1,1}(\Omega)$ – Hilbert space of all elements of $L_2(\Omega)$ whose weak derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$ belong to $L_2(\Omega)$, and scalar product is defined as

$$(u,v) = \int_\Omega \left(uv + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} \right) dxdt$$

$V_2(\Omega)$-solution of the Neumann Problem

Definition 2

The function $u \in V_2(\Omega)$ is called a weak solution of (2)-(5) if

$$0 = \int_0^T \int_0^{s(t)} [a u_x \Phi_x - b u_x \Phi - c u \Phi - u \Phi_t + f \Phi] \, dx \, dt - \\ - \int_0^{s_0} \phi(x) \Phi(x,0) \, dx + \int_0^T g(t) \Phi(0,t) \, dt + \\ + \int_0^T [\gamma(s(t),t) s'(t) - u(s(t),t) s'(t) - \chi(s(t),t)] \Phi(s(t),t) \, dt \qquad (7)$$

for arbitrary $\Phi \in W_2^{1,1}(\Omega)$ such that $\Phi|_{t=T} = 0$.

$$\|u\|_{V_2(\Omega)} = \Big(\operatorname*{ess\,sup}_{0 \le t \le T} \|u(x,t)\|^2_{L_2[0,s(t)]} + \Big\| \frac{\partial u}{\partial x} \Big\|^2_{L_2(\Omega)} \Big)^{\frac{1}{2}}$$

$$\|u\|_{V_2^{1,0}(\Omega)} = \Big(\max_{0 \le t \le T} \|u(x,t)\|^2_{L_2(0,s(t))} + \Big\| \frac{\partial u}{\partial x} \Big\|^2_{L_2(\Omega)} \Big)^{\frac{1}{2}}$$

Formulated optimal control problem will be called Problem $\mathcal{I}$.

Discrete Optimal Control Problem

$$\omega_\tau = \{t_j = j \cdot \tau,\ j = 0, 1, \ldots, n\}$$

$$V_R^n = \{[v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} :\ 0 < \delta \le s_k \le l,$$
$$\max(\|[s]_n\|^2_{w_2^2};\ \|[g]_n\|^2_{w_2^1}) \le R^2\}$$

$$[s]_n = (s_0, s_1, ..., s_n) \in \mathbb{R}^{n+1},\ [g]_n = (g_0, g_1, ..., g_n) \in \mathbb{R}^{n+1}$$

$$\|[s]_n\|^2_{w_2^2} = \sum_{k=0}^{n-1} \tau s_k^2 + \sum_{k=1}^{n} \tau s_{\bar{t},k}^2 + \sum_{k=1}^{n-1} \tau s_{\bar{t}t,k}^2,\ \|[g]_n\|^2_{w_2^1} = \sum_{k=0}^{n-1} \tau g_k^2 + \sum_{k=1}^{n} \tau g_{\bar{t},k}^2.$$

$$s_{\bar{t},k} = \frac{s_k - s_{k-1}}{\tau},\ s_{t,k} = \frac{s_{k+1} - s_k}{\tau},\ s_{\bar{t}t,k} = \frac{s_{k+1} - 2s_k + s_{k-1}}{\tau^2}.$$

Approximation and Interpolation Mappings

Introduce two mappings $\mathcal{Q}_n$ and $\mathcal{P}_n$ between continuous and discrete control sets:

$$\mathcal{Q}_n(v) = [v]_n = ([s]_n, [g]_n), \quad \text{for } v \in V_R$$

where $s_k = s(t_k), g_k = g(t_k), k = 0, 1, ..., n.$

$$\mathcal{P}_n([v]_n) = v^n = (s^n, g^n) \in W_2^2[0,T] \times W_2^1[0,T] \quad \text{for } [v]_n \in V_R^n,$$

$$s^n(t) = \begin{cases} s_0 + \frac{t^2}{2\tau} s_{\bar{t},1} & 0 \le t \le \tau, \\ s_{k-1} + (t - t_{k-1} - \frac{\tau}{2}) s_{\bar{t},k-1} + \frac{1}{2}(t - t_{k-1})^2 s_{\bar{t}t,k-1} & t_{k-1} \le t \le t_k. \end{cases}$$

$$g^n(t) = g_{k-1} + \frac{g_k - g_{k-1}}{\tau}(t - t_{k-1}), \quad t_{k-1} \le t \le t_k, k = \overline{1,n}.$$

Steklov Averages

$$d_k(x) = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} d(x,t)\,dt,\ \ h_k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} h(t)\,dt,$$

d stands for any of the functions a, b, c, f, and h stands for any of the functions ν, μ, g, g^n.
Given $v = (s, g) \in V_R$ define Steklov averages of traces

$$\chi_s^k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \chi(s(t),t)\,dt,\ \ (\gamma_s s')^k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \gamma(s(t),t)s'(t)\,dt.$$

Given $[v]_n = ([s]_n, [g]_n) \in V_R^n$ define Steklov averages of traces

$$\chi_{s^n}^k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \chi(s^n(t),t)\,dt,\ \ (\gamma_{s^n}(s^n)')^k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \gamma(s^n(t),t)(s^n)'(t)\,dt.$$

Discrete State Vector

Definition 3

$[u([v]_n)]_n = (u(x;0), u(x;1), ..., u(x;n))$ is called discrete state vector if

1. $u(x;0) = \phi(x) \in W_2^1[0, s_0]$;
2. $u(x;k) \in W_2^1[0, s_k]$ satisfy the integral identity

$$\int_0^{s_k} \Big(a_k(x) \frac{du(x;k)}{dx} \eta'(x) - b_k \frac{du(x;k)}{dx} \eta(x) - c_k(x) u(x;k) \eta(x) + f_k(x) \eta + u_{\bar{t}}(x;k) \eta(x) \Big) dx + \Big((\gamma_{s^n}(s^n)')^k - \chi_{s^n}^k \Big) \eta(s_k) + g_k^n \eta(0) = 0, \quad \forall \eta \in W_2^1[0, s_k] \qquad (8)$$

3. $u(x;k) \in W_2^1[0, s_k]$ iteratively continued to $[0, l]$ as

$$u(x;k) = u(2^n s_k - x; k), \;\; 2^{n-1} s_k \le x \le 2^n s_k, n = \overline{1, n_k},$$

$$n_k \le N = 1 + \log_2 \left[\frac{l}{\delta} \right] \qquad (9)$$

Discrete Optimal Control Problem

$$\mathcal{I}_n([v]_n) = \beta_0 \tau \sum_{k=1}^{n} \Big(u(0;k) - \nu_k\Big)^2 + \beta_1 \tau \sum_{k=1}^{n} \Big(u(s_k;k) - \mu_k\Big)^2 \quad (10)$$

$$V_R^n = \{[v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} : \ 0 < \delta \le s_k \le l, \\ \max(\|[s]_n\|^2_{w_2^2}; \ \|[g]_n\|^2_{w_2^1}) \le R^2\}$$

$[u([v]_n)]_n = (u(x;0), u(x;1), ..., u(x;n))$ be a discrete state vector.
Formulated discrete optimal control problem will be called Problem $\mathcal{I}_n$.

$$u^\tau(x,t) = u(x;k), \quad \text{if } t_{k-1} < t \le t_k, \ 0 \le x \le l, \ k = \overline{1,n},$$

$$\hat{u}^\tau(x,t) = u(x;k-1) + u_{\bar{t}}(x;k)(t - t_{k-1}), \quad \text{if } t_{k-1} < t \le t_k, \ 0 \le x \le l, \ k = \overline{1,n}$$

$$\hat{u}^\tau(x,t) = u(x;n), \quad \text{if } t \ge T, \ 0 \le x \le l.$$

$$u^\tau \in V_2(D), \quad \hat{u}^\tau \in W_2^{1,1}(D)$$

Existence of the Optimal Control

$$D = \{(x,t): \ 0 < x < l, \ 0 < t \leq T\}$$
$$a, b, c \in L_\infty(D), \ f \in L_2(D),$$
$$\phi \in W_2^1[0, s_0], \ \gamma, \chi \in W_2^{1,1}(D), \ \mu, \nu \in L_2[0,T],$$
$$\frac{\partial a}{\partial x} \in L_\infty(D), \quad \int_0^T \operatorname*{ess\,sup}_{0 \leq x \leq l} \left| \frac{\partial a}{\partial t} \right| dt < +\infty. \tag{11}$$

Theorem 4

The Problem $\mathcal{I}$ has a solution, i.e.

$$V_* = \{v \in V_R : \mathcal{J}(v) = \mathcal{J}_* \equiv \inf_{v \in V_R} \mathcal{J}(v)\} \neq \emptyset$$

Convergence Theorem

Theorem 5

Sequence of discrete optimal control problems $\mathcal{I}_n$ *approximates the optimal control problem* $\mathcal{I}$ *with respect to functional, i.e.*

$$\lim_{n\to+\infty} I_{n_*} = J_*, \tag{12}$$

where

$$I_{n_*} = \inf_{V_R^n} I_n([v]_n),\ n = 1, 2, ...$$

If $[v]_{n_\epsilon} \in V_R^n$ *is chosen such that*

$$I_{n_*} \le I_n([v]_{n_\epsilon}) \le I_{n_*} + \epsilon_n,\ \epsilon_n \downarrow 0,$$

then the sequence $v_n = (s_n, g_n) = \mathcal{P}_n([v]_{n_\epsilon})$ *converges to some element* $v_* = (s_*, g_*) \in V_*$ *weakly in* $W_2^2[0,T] \times W_2^1[0,T]$*, and strongly in* $W_2^1[0,T] \times L_2[0,T]$*. In particular* s_n *converges to* s_* *uniformly on* $[0,T]$*. Moreover, piecewise linear interpolation* $\hat{u}^\tau$ *of the discrete state vector* $[u[v]_{n_\epsilon}]_n$ *converges to the solution* $u(x,t;v_*) \in W_2^{1,1}(\Omega_*)$ *of the Neumann problem (2)-(5) weakly in* $W_2^{1,1}(\Omega_*)$*.*

Preliminary Results

Lemma 6

For sufficiently small time step τ, there exists a unique discrete state vector $[u([v]_n)]_n$ for arbitrary discrete control vector $[v]_n \in V_R^n$.

Lemma 7

For arbitrary sufficiently small $\epsilon > 0$ there exists n_ϵ such that

$$\mathcal{Q}_n(v) \in V_R^n, \quad \text{for all } v \in V_{R-\epsilon} \quad \text{and } n > n_\epsilon. \tag{13}$$

$$\mathcal{P}_n([v]_n) \in V_{R+\epsilon}, \quad \text{for all } [v]_n \in V_R^n \quad \text{and } n > n_\epsilon. \tag{14}$$

Corollary 8

Let either $[v]_n \in V_R^n$ or $[v]_n = \mathcal{Q}_n(v)$ for $v \in V_R$. Then

$$|s_k - s_{k-1}| \leq C\tau, \quad k = 1, 2, \cdots, n \tag{15}$$

where C is independent of n.

Lemma 9

(Tikhonov, Vasil'ev 1980) Sequence I_n approximates the continuous optimal control problem I if and only if the following conditions are satisfied:

1. *For arbitrary sufficiently small $\epsilon > 0$ there exists a number $N_1 = N_1(\epsilon)$ such that $\mathcal{Q}_N(v) \in V_R^n$ for all $v \in V_{R-\epsilon}$ and $N \geq N_1$; and for any fixed $\epsilon > 0$ and for all $v \in V_{R-\epsilon}$ the following inequality is satisfied:*
$$\limsup_{N\to\infty} \Big(I_N(\mathcal{Q}_N(v)) - J(v) \Big) \leq 0. \qquad (16)$$
2. *For arbitrary sufficiently small $\epsilon > 0$ there exists a number $N_2 = N_2(\epsilon)$ such that $\mathcal{P}_N([v]_N) \in V_{R+\epsilon}$ for all $[v]_N \in V_R^N$ and $N \geq N_2$; and for all $[v]_N \in V_R^N$, $N \geq 1$ the following inequality is satisfied:*
$$\limsup_{N\to\infty} \Big(J(\mathcal{P}_N([v]_N)) - I_N([v]_N) \Big) \leq 0. \qquad (17)$$
3. $\limsup_{\epsilon\to 0} J_*(\epsilon) \geq J_*$, $\quad \liminf_{\epsilon\to 0} J_*(-\epsilon) \leq J_*$, *where*
$$J_*(\pm\epsilon) = \inf_{V_{R\pm\epsilon}} J(u)$$

First Energy Estimate and its Consequences

Theorem 10

For all sufficiently small τ discrete state vector $[u([v]_n)]_n$ satisfies:

$$\max_{0\le k\le n}\int_0^l u^2(x;k)\,dx+\tau\sum_{k=1}^{n}\int_0^l\Big|\frac{du(x;k)}{dx}\Big|^2dx\le$$

$$C\Big(\|\phi\|^2_{L_2(0,s_0)}+\|g^n\|^2_{L_2(0,T)}+\|f\|^2_{L_2(D)}+\|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)}$$

$$+\|\chi(s^n(t),t)\|^2_{L_2(0,T)}+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}}u^2(x;k)dx\Big),\qquad (18)$$

$$\max_{0\le k\le n}\int_0^l u^2(x;k)\,dx+\tau\sum_{k=0}^{n}\int_0^l\Big|\frac{du(x;k)}{dx}\Big|^2dx+\tau^2\sum_{k=1}^{n}\int_0^l u_{\bar t}^2(x;k)dx\le$$

$$C\Big(\|\phi\|^2_{W_2^1(0,s_0)}+\|g^n\|^2_{L_2(0,T)}+\|f\|^2_{L_2(D)}+\|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)}$$

$$+\|\chi(s^n(t),t)\|^2_{L_2(0,T)}+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}}u^2(x;k)dx\Big),\qquad (19)$$

Weak Compactness in $V_2^{1,0}$

Theorem 11

Let $[v]_n \in V_R^n, n = 1, 2, ...$ be a sequence of discrete controls and the sequence $\{\mathcal{P}_n([v]_n\}$ converges strongly in $W_2^1[0,T] \times L_2[0,T]$ to $v = (s, g)$. Then the sequence $\{u^\tau\}$ converges as $\tau \to 0$ weakly in $W_2^{1,0}(\Omega)$ to weak solution $u \in V_2^{1,0}(\Omega)$ of the problem (2)-(5), i.e. to the solution of the integral identity (7). Moreover, u satisfies the energy estimate

$$\|u\|^2_{V_2^{1,0}(D)} \leq C\Big(\|\phi\|^2_{L_2(0,s_0)} + \|g\|^2_{L_2(0,T)} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,0}(D)} + \|\chi\|^2_{W_2^{1,0}(D)}\Big) \tag{20}$$

Proof.

$$\tilde{s}^n(t) = s_{k-1} + \frac{s_k - s_{k-1}}{\tau}(t - t_{k-1}), \quad t_{k-1} \leq t \leq t_k, k = \overline{1, n};$$

$$\tilde{s}^n(t) \equiv s_n, \ t \geq T; \ \tilde{s}_1^n(t) = \tilde{s}^n(t + \tau), \quad 0 \leq t \leq T.$$

$$\sup_n \|\tilde{s}_1^n\|_{W_2^1[0,T]} < C_* \tag{21}$$

where C_* is independent of n.

Proof of Weak Compactness in $V_2^{1,0}$

$$\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}}u^2(x;k)dx=\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{t_k}^{t_{k+1}}\left(\tilde{s}^n\right)'(t)\times$$
$$u^2(\tilde{s}^n(t);k)dt=\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{t_k}^{t_{k+1}}\left(\tilde{s}^n\right)'(t)\Big(u^\tau(\tilde{s}^n(t),t-\tau)\Big)^2dt=$$
$$\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{t_{k-1}}^{t_k}\left(\tilde{s}_1^n\right)'(t)\Big(u^\tau(\tilde{s}_1^n(t),t)\Big)^2dt. \qquad (22)$$

$$\Big|\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}}u^2(x;k)dx\Big|\leq\|(\tilde{s}_1^n)'\|_{L_2[0,T]}\|u^\tau(\tilde{s}_1^n(t),t)\|^2_{L_4[0,T]}. \qquad (23)$$

$$\|u(\tilde{s}_1^n(t),t)\|_{L_4[0,T]}\leq\tilde{C}\|u\|_{V_2(D)}, \qquad (24)$$

Proof of Weak Compactness in $V_2^{1,0}$

$$\Big|\sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}} u^2(x;k)dx\Big| \leq C_*\tilde{C}\|u^\tau\|^2_{V_2(D)}. \tag{25}$$

$$C_* < (C\tilde{C})^{-1} \tag{26}$$

$$\|u^\tau\|^2_{V_2^{1,0}(D)} \leq C\Big(\|\phi\|^2_{L_2(0,s_0)} + \|g^n\|^2_{L_2(0,T)} + \|f\|^2_{L_2(D)} + \|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)} + \|\chi(s^n(t),t)\|^2_{L_2(0,T)}\Big), \tag{27}$$

$$\|\gamma(s^n(t),t)(s^n)'(t)\|_{L_2(0,T)} \leq \|(s^n)'\|_{C[0,T]}\|\gamma(s^n(t),t)\|_{L_2[0,T]} \leq C_3\|\gamma\|_{W_2^{1,0}(D)}$$
$$\|\chi(s^n(t),t)\|_{L_2[0,T]} \leq C_3\|\chi\|_{W_2^{1,0}(D)}, \tag{28}$$

$$\|u^\tau\|^2_{V_2^{1,0}(D)} \leq C\Big(\|\phi\|^2_{L_2(0,s_0)}+\|g\|^2_{L_2(0,T)}+\|f\|^2_{L_2(D)}+\|\gamma\|^2_{W_2^{1,0}(D)}+\|\chi\|^2_{W_2^{1,0}(D)}\Big) \tag{29}$$

Proof of Weak Compactness in $V_2^{1,0}$

The sequence $\{u^\tau\}$ is weakly precompact in $W_2^{1,0}(D)$.Let $u \in W_2^{1,0}(D)$ be a weak limit point of u^τ in $W_2^{1,0}(D)$, and assume that whole sequence $\{u^\tau\}$ converges to u weakly in $W_2^{1,0}(D)$. Let us prove that in fact u satisfies the integral identity (7) for arbitrary test function $\Phi \in W_2^{1,1}(\Omega)$ such that $\Phi|_{t=T} = 0$.Due to density of $C^1(\overline{\Omega})$ in $W_2^{1,1}(\Omega)$ it is enough to assume $\Phi \in C^1(\overline{\Omega})$. $\Phi \in C^1(\overline{D}_{T+\tau}),\ \Phi \equiv 0,\quad$ for $T \leq t \leq T+\tau$, where

$$D_{T+\tau} = \{(x,t):\ 0 < x < l,\ 0 < t \leq T+\tau\}$$

$$\Phi(x;k) = \Phi(x,k\tau), \quad \Phi_t(x;k) = \frac{\Phi(x;k+1) - \Phi(x;k)}{\tau}$$

Let Φ^τ, Φ_t^τ be piecewise constant interpolations. Sequences $\{\Phi^\tau\}$, $\left\{\frac{\partial \Phi^\tau}{\partial x}\right\}$ and $\{\Phi_t^\tau\}$ converge uniformly in $\bar{D}$ to Φ, $\frac{\partial \Phi}{\partial x}$ and Φ_t respectively.

Proof of Weak Compactness in $V_2^{1,0}$

Choose $\eta(x) = \tau\Phi(x;k)$ in (8), after summation with respect to $k = \overline{1,n}$ and transformation of the time difference term as follows

$$\tau \sum_{k=1}^{n} \int_0^{s_k} u_{\bar{t}}(x;k)\Phi(x;k)\,dx = -\tau \sum_{k=1}^{n-1} \int_0^{s_{k+1}} u(x;k)\Phi_t(x;k)\,dx -$$

$$\int_0^{s_1} \phi(x)\Phi(x;1)\,dx - \sum_{k=1}^{n-1} \int_{s_k}^{s_{k+1}} u(x;k)\Phi(x;k)\,dx =$$

$$-\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_0^{s_{k+1}} u^\tau \Phi_t^\tau \,dx\,dt - \int_0^{s_1} \phi(x)\Phi^\tau(x,\tau)\,dx$$

$$-\sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} (\tilde{s}^n)'(t) u^\tau(\tilde{s}^n(t), t-\tau)\Phi^\tau(\tilde{s}^n(t), t-\tau)\,dt =$$

$$-\int_0^T \int_0^{s(t)} u^\tau \Phi_t^\tau \,dx\,dt - \int_0^{s_1} \phi(x)\Phi^\tau(x,\tau)\,dx -$$

$$\int_0^{T-\tau} (\tilde{s}_1^n)'(t) u^\tau((\tilde{s}_1^n)(t), t)\Phi^\tau((\tilde{s}_1^n)(t), t)\,dt - \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \int_{s(t)}^{s_{k+1}} u^\tau \Phi_t^\tau \,dx\,dt \quad (20)$$

$$\int_0^T \int_0^{s(t)} \left\{ a\frac{\partial u^\tau}{\partial x}\frac{\partial \Phi^\tau}{\partial x} - b\frac{\partial u^\tau}{\partial x}\Phi^\tau - cu^\tau\Phi^\tau + f\Phi^\tau - u^\tau \Phi^\tau_t \right\} dx\,dt -$$
$$\int_0^{s_0} \phi(x)\Phi^\tau(x,\tau)\,dx - \int_0^{T-\tau} (\tilde{s}_1^n)'(t) u^\tau((\tilde{s}_1^n)(t),t)\Phi^\tau((\tilde{s}_1^n)(t),t)\,dt +$$
$$\int_0^T g^n\Phi^\tau(0,t)\,dt + \int_0^T \Big[\gamma(s^n(t),t)(s^n)'(t) - \chi(s^n(t),t))\Big]\Phi^\tau(s^n(t),t)\,dt = R \tag{31}$$

,

$$R = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{s(t)}^{s_k} \left\{ a\frac{\partial u^\tau}{\partial x}\frac{\partial \Phi^\tau}{\partial x} - b\frac{\partial u^\tau}{\partial x}\Phi^\tau - cu^\tau\Phi^\tau + f\Phi^\tau \right\} dx\,dt -$$
$$\sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \int_{s(t)}^{s_{k+1}} u^\tau \Phi^\tau_t\,dx\,dt + \int_{s_0}^{s_1} \phi(x)\Phi^\tau(x,\tau)\,dx$$
$$+ \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{s^n(t)}^{s_k} \Big[\gamma(s^n(t),t)(s^n)'(t) - \chi(s^n(t),t)\Big]\frac{\partial \Phi^\tau}{\partial x}\,dx\,dt$$

Proof of Weak Compactness in $V_2^{1,0}$

$$\Delta = \bigcup_{k=1}^{n} \{(x,t) : t_{k-1} < t < t_k, \ \min(s(t), s_k) < x < \max(s(t), s_k)\}$$

$$|\Delta| \le \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_{t}^{t_k} |s'(\tau)| \, d\tau \, dt \le \frac{2\sqrt{T}}{3} \|s'\|_{L^2(0,T)} \tau \to 0 \quad \text{as } \tau \to 0$$

$$\lim_{\tau \to 0} R = 0 \tag{32}$$

$$\|u^\tau(\tilde{s}_1^n(t), t) - u^\tau(s(t), t)\|_{L_2[0,T]} \to 0 \quad \text{as } \tau \to 0 \tag{33}$$

$$u^\tau(s(t), t) \to u(s(t), t), \quad \text{weakly in } L_2[0,T] \tag{34}$$

$$\lim_{\tau \to 0} \int_0^{T-\tau} (\tilde{s}_1^n)'(t) u^\tau(\tilde{s}_1^n(t), t) \Phi^\tau(\tilde{s}_1^n(t), t) \, dt = \int_0^T s'(t) u(s(t), t) \Phi(s(t), t) \tag{35}$$

Passing to the limit as $\tau \to 0$, from (21) it follows that u is a weak solution of the problem (2)-(5). Whole sequence $\{u^\tau\}$ converges to $u \in V_2^{1,0}(\Omega)$ weakly in $W_2^{1,0}(\Omega)$. Theorem is proved.

$V_2^{1,0}$ Solvability of the Neumann Problem

Corollary 12

For arbitrary $v = (s, g) \in V_R$ there exists a weak solution $u \in V_2^{1,0}(\Omega)$ of the problem (2)-(5) which satisfy the energy estimate (12):

$$\|u\|^2_{V_2^{1,0}(D)} \le C\Big(\|\phi\|^2_{L_2(0,s_0)} + \|g\|^2_{L_2(0,T)} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,0}(D)} + \|\chi\|^2_{W_2^{1,0}(D)}\Big)$$

Assumptions needed for the weak compactness in $V_2^{1,0}(\Omega)$:

$$\phi \in L_2[0,l],\ \gamma, \chi \in W_2^{1,0}(D),\ a \in L_\infty(D)$$

Second Energy Estimate and its Consequences

Theorem 13

For all sufficiently small τ discrete state vector $[u([v]_n)]_n$ satisfies the following stability estimation:

$$\max_{1\leq k\leq n}\int_0^l \left|\frac{du(x;k)}{dx}\right|^2 dx+\tau\sum_{k=1}^n\int_0^l u_{\bar{t}}^2(x;k)\,dx\leq C\Big[\,\|\phi\|^2_{W_2^1[0,l]}+$$

$$\|g^n\|^2_{W_2^{\frac14}[0,T]}+\|\gamma(s^n(t),t)(s^n)'(t)\|^2_{W_2^{\frac14}[0,T]}+\|\chi(s^n(t),t)\|^2_{W_2^{\frac14}[0,T]}+$$

$$+\|f\|^2_{L_2(D)}+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}}u^2(x;k)dx\Big] \qquad (36)$$

$W_2^{\frac14}[0,T]$ – Banach space of all elements of $L_2[0,T]$ with finite norm

$$\|u\|_{W_2^{\frac14}[0,T]}=\left(\|u\|^2_{L_2[0,T]}+\int_0^T dt\int_0^T\frac{|u(t)-u(\tau)|^2}{|t-\tau|^{\frac32}}d\tau\right)^{\frac12}$$

Lemma 14

Given $[v]_n$ *define the vector function*

$$[\tilde{u}([v]_n)]_n = (\tilde{u}(x;0), \tilde{u}(x;1), ..., \tilde{u}(x;n))$$

$$\tilde{u}(x;k) = \begin{cases} u(x;k) & 0 \le x \le s_k, \\ u(s_k;k) & s_k \le x \le l, k = \overline{0,n}. \end{cases}$$

Then for all sufficiently small τ, $[\tilde{u}([v]_n)]_n$ *satisfies the following estimation:*

$$\max_{1\le k\le n} \int_0^{s_k} \left| \frac{d\tilde{u}(x;k)}{dx} \right|^2 dx + \tau \sum_{k=1}^{m} \int_0^{s_k} \tilde{u}_{\bar{t}}^2(x;k)\, dx +$$
$$\tau^2 \sum_{k=1}^{m} \int_0^{s_k} \left[\left(\frac{d\tilde{u}(x;k)}{dx} \right)_{\bar{t}} \right]^2 dx \le C \Big[\|\phi\|^2_{W_2^1[0,l]} + \|g^n\|^2_{W_2^{\frac{1}{4}}[0,T]}$$
$$+ \|\gamma(s^n(t),t)(s^n)'(t)\|^2_{W_2^{\frac{1}{4}}[0,T]} + \|\chi(s^n(t),t)\|^2_{W_2^{\frac{1}{4}}[0,T]} + \|f\|^2_{L_2(D)}$$
$$+ \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} u^2(x;k)dx \Big], \qquad (37)$$

Proof: Choose $\eta(x) = 2\tau \tilde{u}_{\bar{t}}(x;k)$ in (8) and use the identity

$$2\tau a_k(x)\frac{d\tilde{u}(x;k)}{dx}\left(\frac{d\tilde{u}(x;k)}{dx}\right)_{\bar{t}} = a_k(x)\left(\frac{d\tilde{u}(x;k)}{dx}\right)^2 - a_{k-1}(x)\left(\frac{d\tilde{u}(x;k-1)}{dx}\right.$$

$$-\tau a_{k\bar{t}}(x)\left(\frac{d\tilde{u}(x;k-1)}{dx}\right)^2 + \tau^2 a_k(x)\left[\left(\frac{d\tilde{u}(x;k)}{dx}\right)_{\bar{t}}\right]^2, \qquad (38)$$

$$\int_0^{s_k} a_k(x)\left(\frac{d\tilde{u}(x;k)}{dx}\right)^2 dx - \int_0^{s_{k-1}} a_{k-1}(x)\left(\frac{d\tilde{u}(x;k-1)}{dx}\right)^2 dx +$$

$$2\tau\int_0^{s_k} (\tilde{u}_{\bar{t}}(x;k))^2\,dx + \tau^2\int_0^{s_k} a_k(x)\left[\left(\frac{d\tilde{u}(x;k)}{dx}\right)_{\bar{t}}\right]^2 \leq$$

$$\tau\int_0^{s_k} a_{k\bar{t}}(x)\left(\frac{d\tilde{u}(x;k-1)}{dx}\right)^2 dx + 2\tau\int_0^{s_k} b_k(x)\frac{d\tilde{u}(x;k)}{dx}\tilde{u}_{\bar{t}}(x;k)\,dx$$

$$+2\tau\int_0^{s_k} c_k(x)\tilde{u}(x;k)\tilde{u}_{\bar{t}}(x;k)\,dx - 2\tau\int_0^{s_k} f_k(x)u_{\bar{t}}(x;k)\,dx$$

$$-2\tau\left[(\gamma_{s^n}(s^n)')^k - \chi^k_{s^n}\right]\tilde{u}_{\bar{t}}(s_k;k) - 2\tau g^n_k \tilde{u}_{\bar{t}}(0;k) \qquad (39)$$

By adding inequalities (39) with respect to k from 1 to arbitrary $m \leq n$ we derive

$$\int_0^{s_m} a_m(x) \left(\frac{d\tilde{u}(x;m)}{dx} \right)^2 dx + 2\tau \sum_{k=1}^{m} \int_0^{s_k} \tilde{u}_{\bar{t}}^2(x;k)\, dx +$$

$$\tau^2 \sum_{k=1}^{m} \int_0^{s_k} a_k(x) \left[\left(\frac{d\tilde{u}(x;k)}{dx} \right)_{\bar{t}} \right]^2 dx \leq \tau \sum_{k=1}^{m} \int_0^{s_k} a_{k\bar{t}}(x) \left(\frac{d\tilde{u}(x;k-1)}{dx} \right)^2 dx$$

$$+2\tau \sum_{k=1}^{m} \int_0^{s_k} b_k(x) \frac{d\tilde{u}(x;k)}{dx} \tilde{u}_{\bar{t}}(x;k)\, dx + 2\tau \sum_{k=1}^{m} \int_0^{s_k} c_k(x) \tilde{u}(x;k) \tilde{u}_{\bar{t}}(x;k)\, dx$$

$$-2\tau \sum_{k=1}^{m} \int_0^{s_k} f_k(x) \tilde{u}_{\bar{t}}(x;k)\, dx + \int_0^{s_0} a_0(x) \left(\frac{d\phi}{dx} \right)^2 dx -$$

$$2\tau \sum_{k=1}^{m} \left[(\gamma_{s^n} (s^n)')^k - \chi_{s^n}^k \right] \tilde{u}_{\bar{t}}(s_k;k) - 2\tau \sum_{k=1}^{m} g_k^n \tilde{u}_{\bar{t}}(0;k) \qquad (40)$$

$$a_0 \int_0^{s_m} \left| \frac{d\tilde{u}(x;k)}{dx} \right|^2 dx + \tau \sum_{k=1}^{m} \int_0^{s_k} \tilde{u}_{\bar{t}}^2(x;k)\,dx +$$
$$a_0 \tau^2 \sum_{k=1}^{m} \int_0^{s_k} \left[\left(\frac{d\tilde{u}(x;k)}{dx} \right)_{\bar{t}} \right]^2 dx \le C\tau \sum_{k=1}^{m} \Big[\int_0^{s_k} u^2(x;k)\,dx +$$
$$\int_0^{s_k} \left| \frac{du(x;k)}{dx} \right|^2 dx + \int_0^{s_k} f_k^2(x)\,dx \Big] C \int_0^{s_0} \left| \frac{d\phi}{dx} \right|^2 dx -$$
$$2\tau \sum_{k=1}^{m} \left[(\gamma_{s^n}(s^n)')^k - \chi_{s^n}^k \right] \tilde{u}_{\bar{t}}(s_k;k) - 2\tau \sum_{k=1}^{m} g_k^n \tilde{u}_{\bar{t}}(0;k)\,dx \qquad (41)$$

$$\|\gamma(s^n(t),t)\|_{W_2^{\frac{1}{4}}[0,T]} \le C\|\gamma\|_{W_2^{1,1}(D)}, \ \|\chi(s^n(t),t)\|_{W_2^{\frac{1}{4}}[0,T]} \le C\|\chi\|_{W_2^{1,1}(D)}, \qquad (42)$$

$$\|\gamma(s^n(t),t)(s^n)'(t)\|_{W_2^{\frac{1}{4}}[0,T]} \le C_1 \|\gamma(s^n(t),t)\|_{W_2^{\frac{1}{4}}[0,T]} \|s^n\|_{W_2^2[0,T]}$$
$$\le C\|\gamma\|_{W_2^{1,1}(D)}, \qquad (43)$$

Let $w(x,t)$ be a function in $W_2^{2,1}(D)$ such that

$$w(x,0) = \phi(x) \quad \text{for } x \in [0,s_0],\ a(0,t)w_x(0,t) = g^n(t), \qquad \text{for a.e. } t \in [0,T] \tag{44}$$

$$a(s^n(t),t)w_x(s^n(t),t) = \gamma(s^n(t),t)(s^n)'(t) - \chi(s^n(t),t) \quad \text{for a.e. } t \in [0,T] \tag{45}$$

$$\|w\|_{W_2^{2,1}(D)} \leq C\Big[\|g^n\|_{W_2^{\frac{1}{4}}[0,T]} + \|\phi(x)\|_{W_2^1[0,s_0]} + \|\gamma(s^n(t),t)(s^n)'(t) - \chi(s^n(t),t)\|_{W_2^{\frac{1}{4}}[0,T]}\Big] \tag{46}$$

w can be constructed as a solution from $W_2^{2,1}(\Omega^n)$ of the heat eq. in

$$\Omega^n = \{0 < x < s^n(t), 0 < t < T\}.$$

By replacing in original Neumann problem $s \to s^n, g \to g^n, u \to u - w$ derive (41) without last three terms and with $f_k(x)$ replaced by Steklov average $F_k(x)$ of

$$F = f + w_t - (aw_x)_x - bw_x - cw \in L_2(D). \tag{47}$$

From modified (41),(46),(47), desired (37) follows. Lemma is proved.

In the next lemma we prove the stability estimation with l being replaced by s_k.

Lemma 15

For all sufficiently small τ, discrete state vector $[u([v]_n)]_n$ satisfies the following estimation:

$$\max_{1\leq k\leq n}\int_0^{s_k}\left|\frac{du(x;k)}{dx}\right|^2 dx+\tau\sum_{k=1}^{n}\int_0^{s_k}u_{\bar{t}}^2(x;k)\,dx\leq$$
$$C\Big[\|\phi\|^2_{W_2^1[0,l]}+\|g^n\|^2_{W_2^{\frac{1}{4}}[0,T]}+\|\gamma(s^n(t),t)(s^n)'(t)\|^2_{W_2^{\frac{1}{4}}[0,T]}+$$
$$\|\chi(s^n(t),t)\|^2_{W_2^{\frac{1}{4}}[0,T]}+\|f\|^2_{L_2(D)}+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}}u^2(x;k)dx\Big],\tag{48}$$

we only need to estimate $\int\limits_0^{s_k} u_{\bar{t}}^2(x;k)\,dx,\ \ s_{k-1} < s_k.$

$$\int\limits_0^{s_k} u_{\bar{t}}^2(x;k)dx = \int\limits_0^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x;k)dx + \int\limits_{s_{k-1}}^{s_k} u_{\bar{t}}^2(x;k)dx$$

$$\int\limits_{s_{k-1}}^{s_k} \Big|\frac{u(x;k)-u(x;k-1)}{\tau}\Big|^2 dx \leq 2\int\limits_{s_{k-1}}^{s_k} \Big|\frac{u(x;k)-u(2s_{k-1}-x;k)}{\tau}\Big|^2 dx$$

$$+2\int\limits_{s_{k-1}}^{s_k} \Big|\frac{u(2s_{k-1}-x;k)-u(2s_{k-1}-x;k-1)}{\tau}\Big|^2 dx \leq$$

$$2\int\limits_{s_{k-1}}^{s_k} \Big|\frac{1}{\tau}\int\limits_{2s_{k-1}-x}^{x} \frac{du(y;k)}{dy}\Big|^2 dx + 2\int\limits_{s_{k-1}-(s_k-s_{k-1})}^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x;k)dx \leq$$

$$\frac{2}{\tau^2}\int\limits_{s_{k-1}}^{s_k}\int\limits_{2s_{k-1}-x}^{x} \Big|\frac{du(y;k)}{dy}\Big|^2 dy 2(x-s_{k-1})dx + 2\int\limits_{s_{k-1}-C\tau}^{s_{k-1}} \tilde{u}_{\bar{t}}^2(x;k)dx \leq$$

$$2\int_{s_{k-1}-C\tau}^{s_k}\left|\frac{d\tilde{u}(x;k)}{dx}\right|^2 dx+2\int_{s_{k-1}-C\tau}^{s_{k-1}}\tilde{u}^2_{\bar{t}}(x;k)dx. \tag{49}$$

Hence, for sufficiently small τ we have

$$\int_0^{s_k} u^2_{\bar{t}}(x;k)dx \le 2\int_{s_{k-1}-C\tau}^{s_k}\left|\frac{d\tilde{u}(x;k)}{dx}\right|^2 dx+\int_0^{s_{k-1}}\tilde{u}^2_{\bar{t}}(x;k)dx$$

$$+2\int_{s_{k-1}-C\tau}^{s_{k-1}}\tilde{u}^2_{\bar{t}}(x;k)dx \le 2\int_0^{s_k}\left|\frac{d\tilde{u}(x;k)}{dx}\right|^2 dx+3\int_0^{s_{k-1}}\tilde{u}^2_{\bar{t}}(x;k)dx. \tag{50}$$

Lemma is proved. By using extension lemma, from Lemma 15 second energy estimate follows. Theorem is proved.

Weak Compactness in $W_2^{1,1}$

Theorem 16

Let $[v]_n \in V_R^n, n = 1, 2, \ldots$ be a sequence of discrete controls and the sequence $\{\mathcal{P}_n([v]_n\}$ converges weakly in $W_2^2[0,T] \times W_2^1[0,T]$ to $v = (s,g)$. Then the sequence $\{\hat{u}^\tau\}$ converges as $\tau \to 0$ weakly in $W_2^{1,1}(\Omega)$ to weak solution $u \in W_2^{1,1}(\Omega)$ of the problem (2)-(5), i.e. to the solution of the integral identity (6). Moreover, u satisfies the energy estimate

$$\|u\|^2_{W_2^{1,1}(D)} \leq C\Big(\|\phi\|^2_{W_2^1(0,s_0)} + \|g\|^2_{W_2^{\frac{1}{4}}} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,1}(D)} + \|\chi\|^2_{W_2^{1,1}(D)}\Big) \tag{51}$$

Corollary 17

For arbitrary $v = (s,g) \in V_R$ there exists a weak solution $u \in W_2^{1,1}(\Omega)$ of the problem (2)-(5) which satisfy the energy estimate

$$\|u\|^2_{W_2^{1,1}(D)} \leq C\Big(\|\phi\|^2_{W_2^1(0,s_0)} + \|g\|^2_{W_2^{\frac{1}{4}}} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,1}(D)} + \|\chi\|^2_{W_2^{1,1}(D)}\Big)$$