Optimal Control and Inverse Problems for PDEs.
I. Inverse Stefan Problem

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Introduction

One-phase Stefan problem: find the temperature function $u(x,t)$ in $\Omega = \{(x,t): \ 0 < x < s(t), \ 0 < t \leq T\}$ and the free boundary $x = s(t)$ from (1)-(5):

$$(a(x,t)u_x)_x + b(x,t)u_x + c(x,t)u - u_t = f(x,t), \quad \text{for } (x,t) \in \Omega \quad (1)$$

$$u(x,0) = \phi(x), \qquad 0 \leq x \leq s(0) = s_0 \quad (2)$$

$$a(0,t)u_x(0,t) = g(t), \qquad 0 \leq t \leq T \quad (3)$$

$$a(s(t),t)u_x(s(t),t) + \gamma(s(t),t)s'(t) = \chi(s(t),t), \qquad 0 \leq t \leq T \quad (4)$$

$$u(s(t),t) = \mu(t), \qquad 0 \leq t \leq T \quad (5)$$

$$a(x,t) \geq a_0 > 0, \quad s_0 > 0 \quad (6)$$

Inverse Stefan Problem

Assume that the boundary heat flux $g(t)$ is not available.

$$u(0,t) = \nu(t), \quad \text{for } 0 \le t \le T \tag{7}$$

Inverse Stefan Problem (ISP): *Find the functions $u(x,t)$ and $s(t)$ and the boundary heat flux $g(t)$ satisfying conditions (1)-(7).*

$$\begin{gathered}
(a(x,t)u_x)_x + b(x,t)u_x + c(x,t)u - u_t = f(x,t), \quad \text{for } (x,t) \in \Omega \\
u(x,0) = \phi(x), \qquad 0 \le x \le s(0) = s_0 \\
a(0,t)u_x(0,t) = g(t), \qquad 0 \le t \le T \\
a(s(t),t)u_x(s(t),t) + \gamma(s(t),t)s'(t) = \chi(s(t),t), \qquad 0 \le t \le T \\
u(s(t),t) = \mu(t), \qquad 0 \le t \le T
\end{gathered}$$

$$u(x,T) = w(x), \quad \text{for } 0 \le x \le s(T),$$

Optimal Control Problem

$$\mathcal{J}(v) = \beta_0 \|u(0,t) - \nu(t)\|^2_{L_2[0,T]} + \beta_1 \|u(s(t),t) - \mu(t)\|^2_{L_2[0,T]} \qquad (8)$$

$$V_R = \Big\{ v = (s,g) \in W_2^2[0,T] \times W_2^1[0,T] : \delta \le s(t) \le l, \\ s(0) = s_0, \max(\ \|s\|_{W_2^2}; \ \|g\|_{W_2^1}) \le R \Big\}$$

$$(a(x,t)u_x)_x + b(x,t)u_x + c(x,t)u - u_t = f(x,t), \quad \text{for } (x,t) \in \Omega \qquad (9)$$

$$u(x,0) = \phi(x), \qquad 0 \le x \le s(0) = s_0 \qquad (10)$$

$$a(0,t)u_x(0,t) = g(t), \qquad 0 \le t \le T \qquad (11)$$

$$a(s(t),t)u_x(s(t),t) + \gamma(s(t),t)s'(t) = \chi(s(t),t), \qquad 0 \le t \le T \qquad (12)$$

$$\Omega = \{(x,t) : \ 0 < x < s(t), \ 0 < t \le T\}$$

$W_2^{1,1}(\Omega)$-solution of the Neumann Problem

Definition 1

The function $u \in W_2^{1,1}(\Omega)$ is called a weak solution of the problem (9)-(12) if $u(x,0) = \phi(x) \in W_2^1[0, s_0]$ and

$$0 = \int_0^T \int_0^{s(t)} [au_x \Phi_x - bu_x \Phi - cu\Phi + u_t \Phi + f\Phi]\, dx\, dt$$
$$+ \int_0^T [\gamma(s(t),t)s'(t) - \chi(s(t),t)]\Phi(s(t),t)\, dt + \int_0^T g(t)\Phi(0,t)\, dt \quad (13)$$

for arbitrary $\Phi \in W_2^{1,1}(\Omega)$

$W_2^{1,1}(\Omega)$ – Hilbert space of all elements of $L_2(\Omega)$ whose weak derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$ belong to $L_2(\Omega)$, and scalar product is defined as

$$(u,v) = \int_\Omega \left(uv + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} \right) dxdt$$

$V_2(\Omega)$-solution of the Neumann Problem

Definition 2

The function $u \in V_2(\Omega)$ is called a weak solution of (9)-(12) if

$$0 = \int_0^T \int_0^{s(t)} [a u_x \Phi_x - b u_x \Phi - c u \Phi - u \Phi_t + f \Phi] \, dx \, dt -$$

$$- \int_0^{s_0} \phi(x) \Phi(x,0) \, dx + \int_0^T g(t) \Phi(0,t) \, dt +$$

$$+ \int_0^T [\gamma(s(t),t) s'(t) - u(s(t),t) s'(t) - \chi(s(t),t)] \Phi(s(t),t) \, dt \qquad (14)$$

for arbitrary $\Phi \in W_2^{1,1}(\Omega)$ such that $\Phi|_{t=T} = 0$.

$$\|u\|_{V_2(\Omega)} = \Big(\operatorname*{ess\ sup}_{0 \le t \le T} \|u(x,t)\|^2_{L_2[0,s(t)]} + \Big\| \frac{\partial u}{\partial x} \Big\|^2_{L_2(\Omega)} \Big)^{\frac{1}{2}}$$

$$\|u\|_{V_2^{1,0}(\Omega)} = \Big(\max_{0 \le t \le T} \|u(x,t)\|^2_{L_2(0,s(t))} + \Big\| \frac{\partial u}{\partial x} \Big\|^2_{L_2(\Omega)} \Big)^{\frac{1}{2}}$$

Formulated optimal control problem will be called Problem $\mathcal{I}$.

Discrete Optimal Control Problem

$$\omega_\tau = \{t_j = j \cdot \tau,\ j = 0, 1, \ldots, n\}$$

$$V_R^n = \{[v]_n = ([s]_n, [g]_n) \in \mathbb{R}^{2n+2} :\ 0 < \delta \leq s_k \leq l, \\ \max(\|[s]_n\|^2_{w_2^2};\ \|[g]_n\|^2_{w_2^1}) \leq R^2\}$$

$$[s]_n = (s_0, s_1, ..., s_n) \in \mathbb{R}^{n+1},\ [g]_n = (g_0, g_1, ..., g_n) \in \mathbb{R}^{n+1}$$

$$\|[s]_n\|^2_{w_2^2} = \sum_{k=0}^{n-1} \tau s_k^2 + \sum_{k=1}^{n} \tau s_{\bar{t},k}^2 + \sum_{k=1}^{n-1} \tau s_{\bar{t}t,k}^2,\ \|[g]_n\|^2_{w_2^1} = \sum_{k=0}^{n-1} \tau g_k^2 + \sum_{k=1}^{n} \tau g_{\bar{t},k}^2.$$

$$s_{\bar{t},k} = \frac{s_k - s_{k-1}}{\tau},\ s_{t,k} = \frac{s_{k+1} - s_k}{\tau},\ s_{\bar{t}t,k} = \frac{s_{k+1} - 2s_k + s_{k-1}}{\tau^2}.$$

Approximation and Interpolation Mappings

Introduce two mappings $\mathcal{Q}_n$ and $\mathcal{P}_n$ between continuous and discrete control sets:

$$\mathcal{Q}_n(v) = [v]_n = ([s]_n, [g]_n), \quad \text{for } v \in V_R$$

where $s_k = s(t_k), g_k = g(t_k), k = 0, 1, ..., n.$

$$\mathcal{P}_n([v]_n) = v^n = (s^n, g^n) \in W_2^2[0,T] \times W_2^1[0,T] \quad \text{for } [v]_n \in V_R^n,$$

$$s^n(t) = \begin{cases} s_0 + \frac{t^2}{2\tau} s_{\bar{t},1} & 0 \le t \le \tau, \\ s_{k-1} + (t - t_{k-1} - \frac{\tau}{2}) s_{\bar{t},k-1} + \frac{1}{2}(t - t_{k-1})^2 s_{\bar{t}t,k-1} & t_{k-1} \le t \le t_k. \end{cases}$$

$$g^n(t) = g_{k-1} + \frac{g_k - g_{k-1}}{\tau}(t - t_{k-1}), \quad t_{k-1} \le t \le t_k, k = \overline{1, n}.$$

Steklov Averages

$$d_k(x) = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} d(x,t)\,dt,\;\; h_k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} h(t)\,dt,$$

d stands for any of the functions a, b, c, f, and h stands for any of the functions ν, μ, g, g^n.
Given $v = (s, g) \in V_R$ define Steklov averages of traces

$$\chi_s^k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \chi(s(t),t)\,dt,\;\; (\gamma_s s')^k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \gamma(s(t),t)s'(t)\,dt.$$

Given $[v]_n = ([s]_n, [g]_n) \in V_R^n$ define Steklov averages of traces

$$\chi_{s^n}^k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \chi(s^n(t),t)\,dt,\;\; (\gamma_{s^n}(s^n)')^k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \gamma(s^n(t),t)(s^n)'(t)\,dt.$$

Discrete State Vector

Definition 3

$[u([v]_n)]_n = (u(x;0), u(x;1), ..., u(x;n))$ is called discrete state vector if

1. $u(x;0) = \phi(x) \in W_2^1[0, s_0]$;
2. $u(x;k) \in W_2^1[0, s_k]$ satisfy the integral identity

$$\int_0^{s_k} \Big(a_k(x) \frac{du(x;k)}{dx} \eta'(x) - b_k \frac{du(x;k)}{dx} \eta(x) - c_k(x) u(x;k) \eta(x) + f_k(x) \eta + u_{\bar{t}}(x;k)\eta(x) \Big) dx + \Big((\gamma_{s^n}(s^n)')^k - \chi_{s^n}^k \Big) \eta(s_k) + g_k^n \eta(0) = 0, \quad \forall \eta \in W_2^1[0, s_k] \tag{15}$$

3. $u(x;k) \in W_2^1[0, s_k]$ iteratively continued to $[0, l]$ as

$$u(x;k) = u(2^n s_k - x; k), \; 2^{n-1} s_k \le x \le 2^n s_k, n = \overline{1, n_k},$$

$$n_k \le N = 1 + \log_2 \left[\frac{l}{\delta} \right] \tag{16}$$

Discrete Optimal Control Problem

$$\mathcal{I}_n([v]_n) = \beta_0\tau\sum_{k=1}^{n}\Big(u(0;k)-\nu_k\Big)^2 + \beta_1\tau\sum_{k=1}^{n}\Big(u(s_k;k)-\mu_k\Big)^2 \quad (17)$$

$$V_R^n = \{[v]_n = ([s]_n,[g]_n)\in\mathbb{R}^{2n+2}:\ 0<\delta\le s_k\le l,$$
$$\max(\|[s]_n\|^2_{w_2^2};\ \|[g]_n\|^2_{w_2^1})\le R^2\}$$

$[u([v]_n)]_n = (u(x;0),u(x;1),...,u(x;n))$ be a discrete state vector. Formulated discrete optimal control problem will be called Problem $\mathcal{I}_n$.

$$u^\tau(x,t) = u(x;k), \quad \text{if } t_{k-1}<t\le t_k,\ 0\le x\le l,\ k=\overline{1,n},$$

$$\hat{u}^\tau(x,t) = u(x;k-1)+u_{\bar{t}}(x;k)(t-t_{k-1}), \quad \text{if } t_{k-1}<t\le t_k,\ 0\le x\le l,\ k=\overline{1,n}$$

$$\hat{u}^\tau(x,t) = u(x;n), \quad \text{if } t\ge T,\ 0\le x\le l.$$

$$u^\tau\in V_2(D),\quad \hat{u}^\tau\in W_2^{1,1}(D)$$

Existence of the Optimal Control

$$D = \{(x,t) : \ 0 < x < l, \ 0 < t \leq T\}$$

$$a, b, c \in L_\infty(D), \ f \in L_2(D),$$

$$\phi \in W_2^1[0, s_0], \ \gamma, \chi \in W_2^{1,1}(D), \ \mu, \nu \in L_2[0,T],$$

$$\frac{\partial a}{\partial x} \in L_\infty(D), \quad \int_0^T \operatorname*{ess\ sup}_{0 \leq x \leq l} \left| \frac{\partial a}{\partial t} \right| dt < +\infty. \tag{18}$$

Theorem 4

The Problem $\mathcal{I}$ has a solution, i.e.

$$V_* = \{v \in V_R : \mathcal{J}(v) = \mathcal{J}_* \equiv \inf_{v \in V_R} \mathcal{J}(v)\} \neq \emptyset$$

Convergence Theorem

Theorem 5

Sequence of discrete optimal control problems $\mathcal{I}_n$ approximates the optimal control problem $\mathcal{I}$ with respect to functional, i.e.

$$\lim_{n\to+\infty} I_{n_*} = J_*, \tag{19}$$

where

$$I_{n_*} = \inf_{V_R^n} I_n([v]_n),\ n = 1, 2, ...$$

If $[v]_{n_\epsilon} \in V_R^n$ is chosen such that

$$I_{n_*} \leq I_n([v]_{n_\epsilon}) \leq I_{n_*} + \epsilon_n,\ \epsilon_n \downarrow 0,$$

then the sequence $v_n = (s_n, g_n) = \mathcal{P}_n([v]_{n_\epsilon})$ converges to some element $v_* = (s_*, g_*) \in V_*$ weakly in $W_2^2[0,T] \times W_2^1[0,T]$, and strongly in $W_2^1[0,T] \times L_2[0,T]$. In particular s_n converges to s_* uniformly on $[0,T]$. Moreover, piecewise linear interpolation $\hat{u}^\tau$ of the discrete state vector $[u[v]_{n_\epsilon}]_n$ converges to the solution $u(x,t;v_*) \in W_2^{1,1}(\Omega_*)$ of the Neumann problem (1)-(4) weakly in $W_2^{1,1}(\Omega_*)$.

Lemma 6

For sufficiently small time step τ, there exists a unique discrete state vector $[u([v]_n)]_n$ for arbitrary discrete control vector $[v]_n \in V_R^n$.

The *Proof* uses a Sobolev energy estimate and Galerkin method.
1) *Uniqueness*: Assume

$$u(x;k-1) \equiv 0, \;\; f_k(x) \equiv 0, \;\; \left(\gamma_{s^n}(s^n)'\right)^k = \chi_{s^n}^k = g_k^n = 0$$

We need to show $u(x;k)$ solves (15) $\Rightarrow u(x;k) \equiv 0$.Choose $\eta(x) = u(x;k)$ in (15):

$$\int_0^{s_k} \left[a_k(x) \left(\frac{du}{dx} \right)^2 - b_k \frac{du}{dx} u - c_k u^2 + \frac{1}{\tau} u^2 \right] dx = 0$$

$$a_0 \int_0^{s_k} \left(\frac{du}{dx} \right)^2 dx + \frac{1}{\tau} \int_0^{s_k} u^2 \, dx \leq$$
$$\leq \frac{\epsilon M}{2} \int_0^{s_k} \left(\frac{du}{dx} \right)^2 dx + \left(\frac{M}{2\epsilon} + M \right) \int_0^{s_k} u^2 \, dx$$

Preliminary Results

$\epsilon = a_0/M \Rightarrow$

$$\frac{a_0}{2}\int_0^{s_k}\left(\frac{du}{dx}\right)^2 dx + \left(\frac{1}{\tau}-\frac{1}{\tau_0}\right)\int_0^{s_k} u^2\,dx \le 0, \quad \tau_0 = \left(\frac{M^2}{2a_0}+M\right)^{-1}$$

$u(x;k) \equiv 0$, $\tau < \tau_0$.
2) *Existence*: Consider

$$u_N(x) = \sum_{i=1}^{N} d_i \psi_i(x), \quad \{\psi_i\} \subseteq W_2^1[0, s_k]$$

$$\int_0^{s_k}\Big[a_k(x)\frac{du_N}{dx}\psi_i'(x) - b_k(x)\frac{du_N}{dx}\psi_i(x) - c_k(x)u_N(x)\psi_i(x)$$
$$+\frac{1}{\tau}u_N(x)\psi_i(x) + f_k(x)\psi_i(x)\Big]\,dx = \frac{1}{\tau}\int_0^{s_k} u(x;k-1)\psi_i\,dx -$$
$$\Big[\big(\gamma_{s^n}(s^n)'\big)^k - \chi_{s^n}^k\Big]\psi_i(s_k) - g^k\psi_i(0),\ \ i = 1,\dots,N$$

Preliminary Results

Corresponding homogeneous system:

$$\sum_{j=1}^{N} d_j \int_0^{s_k} \Big[a_k(x)\psi_j'(x)\psi_i'(x) - b_k(x)\psi_j'(x)\psi_i(x) - c_k(x)\psi_j(x)\psi_i(x) + \frac{1}{\tau}\psi_j(x)\psi_i(x) + f_k(x)\psi_i(x) \Big] dx = 0, \ i = 1, \dots, N$$

Multiply by d_i and add to derive

$$\int_0^{s_k} \left[a_k(x) \left(\frac{du_N}{dx} \right)^2 - b_k(x) \frac{du_N}{dx} u_N(x) - c_k(x) u_N^2(x) + \frac{1}{\tau} u_N^2(x) \right] dx = 0, \ i = 1, \dots, N$$

In a similar way to the uniqueness result, it follows that $u_N \equiv 0$, which implies the unique solution $u_N(x)$ to the original system.

Preliminary Results

We now seek a uniform estimation of sequence $\{u_N(x)\}$ in order to pass $N \to \infty$ and find the corresponding discrete state vector. Multiply the system by d_i and add with respect to i to derive

$$\int_0^{s_k} \left[a_k(x) \left(\frac{du_N}{dx} \right)^2 - b_k(x) \frac{du_N}{dx} u_N(x) - c_k(x) u_N^2(x) \right.$$
$$\left. + \frac{1}{\tau} u_N^2(x) + f_k(x) u_N(x) \right] dx = \frac{1}{\tau} \int_0^{s_k} u(x; k-1) u_N(x)\, dx -$$
$$- \left[\left(\gamma_{s^n} (s^n)' \right)^k - \chi_{s^n}^k \right] u_N(s_k) - g^k u_N(0)$$

Estimate the integrals on the LHS and estimate as before to find

$$\frac{a_0}{2} \int_0^{s_k} \left(\frac{du_N}{dx} \right)^2 + \frac{1}{2\tau} \int_0^{s_k} u_N^2(x) \leq \left| \left| \left(\gamma_{s^n} (s^n)' \right)^k \right| + |\chi_{s^n}^k| \right| |u_N(s_k)| +$$
$$+ |g^k| |u_N(0)| + \frac{1}{\tau} \int_0^{s_k} \left[|f_k(x)| + \frac{1}{\tau} |u(x; k-1)| \right] |u_N(x)|\, dx$$

$\tau \leq \frac{\tau_0}{2}$. Morrey's inequality implies

$$\max\{|u_N(0)|; |u_N(s_k)|\} \leq \|u_N\|_{C[0,s_k]} \leq C \|u_N\|_{W_2^1[0,s_k]}$$

Preliminary Results

Using Cauchy inequality with ϵ, it follows that

$$\|u_N\|^2_{W_2^1[0,s_k]} \leq C\Big(\|u(x;k-1)\|^2_{L_2[0,s_k]} + \|f_k\|^2_{L_2[0,s_k]} + \\ + \Big|\Big(\gamma_{s^n}(s^n)'\Big)^k\Big|^2 + \big|\chi^k_{s^n}\big|^2 + |g^k|^2\Big)$$

Hence $\{u_N\}$ is weakly precompact in $W_2^1[0,s_k]$. Let $v(x)$ be the weak limit point. Pass to the limit in the integral identity for u_N to find

$$\int_0^{s_k}\Big[a_k(x)\frac{dv}{dx}\psi_i'(x) - b_k(x)\frac{dv}{dx}\psi_i(x) - c_k(x)v(x)\psi_i(x) \\ +\frac{1}{\tau}v(x)\psi_i(x) + f_k(x)\psi_i(x)\Big]\,dx = \frac{1}{\tau}\int_0^{s_k} u(x;k-1)\psi_i\,dx - \\ \Big[\Big(\gamma_{s^n}(s^n)'\Big)^k - \chi^k_{s^n}\Big]\psi_i(s_k) - g^k\psi_i(0),\ i = 1,2,\dots$$

That is, v satisfies the appropriate integral identity with $\eta(x) = \psi_i(x)$. Since $\{\psi_i\}$ is fundamental in $W_2^1[0,s_k]$, it follows that $v(x)$ satisfies the integral identity for any $\eta(x) \in W_2^1[0,s_k]$. By uniqueness, $u_N(x) \to v(x)$ weakly in $W_2^1[0,s_k]$. Lemma is proved.

Lemma 7

For arbitrary sufficiently small $\epsilon > 0$ there exists n_ϵ such that

$$\mathcal{Q}_n(v) \in V_R^n, \quad \textit{for all } v \in V_{R-\epsilon} \quad \textit{and } n > n_\epsilon. \tag{20}$$

$$\mathcal{P}_n([v]_n) \in V_{R+\epsilon}, \quad \textit{for all } [v]_n \in V_R^n \quad \textit{and } n > n_\epsilon. \tag{21}$$

The *Proof* follows from calculation of the corresponding norms.
1) Let $0 < \epsilon \ll R$, $v \in V_{R-\epsilon}$, $Q(b) = [v]_n = \big([s]_n, [g]_n\big)$.

$$\sum_{k=1}^{n-1} \tau s_{\bar{t}t,k}^2 = \sum_{k=1}^{n-1} \frac{1}{\tau^3} \Big[\int_{t_k}^{t_{k+1}} \big(s'(t) - s'(t-\tau)\, dt\big)^2 \Big] \le$$

$$\le \frac{1}{\tau^2} \int_\tau^T |s'(t) - s'(t-\tau)|\, dt \le \frac{1}{\tau} \int_\tau^T \int_{t-\tau}^t |s''(\xi)|^2 \, d\xi\, dt$$

so

$$\sum_{k=1}^{n-1} \tau s_{\bar{t}t,k}^2 \le \int_0^T |s''(t)|^2 \, dt, \text{ similarly, } \sum_{k=1}^{n} \tau s_{\bar{t},k}^2 \le \int_0^T |s'(t)|^2 \, dt$$

Preliminary Results

Estimating the last term gives

$$\left|\sum_{k=0}^{n-1}\tau s_k^2-\int_0^T s^2(t)\,dt\right| = \left|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\int_t^{t_k}\left(s^2(\xi)\right)'\,d\xi\,dt\right| \leq$$

$$\leq \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t}\left[s^2(\xi)+\left(s'(\xi)\right)^2\right]\,d\xi\,dt \leq \tau\int_0^T\left[s^2(t)+\left(s'(t)\right)^2\right]\,dt \leq R^2\tau$$

similarly,

$$\sum_{k=1}^{n}\tau g_{\bar{t},k}^2 \leq \int_0^T |g'(t)|^2\,dt, \quad \left|\sum_{k=0}^{n-1}\tau g_k^2-\int_0^T g^2(t)\,dt\right| \leq R^2\tau$$

Hence

$$\max\left(\|[s]_n\|^2_{w_2^2};\|[g]_n\|^2_{w_2^1}\right) \leq \max\left(\|s\|^2_{W_2^2[0,T]};\|g\|^2_{W_2^1[0,T]}\right)+R^2\tau \leq$$
$$\leq (R-\epsilon)^2+R^2\tau \leq R^2$$

When $n \geq n_\epsilon = \lceil\frac{RT}{\epsilon}\rceil+1$. Part 1 of the lemma, equation (20), is proved.

2) Let $[v]_n \in V_R^n$, and assume $v = (s,g) = P_n([v]_n)$. Calculating directly,

$$\|s\|^2_{W_2^2} \le \sum_{k=0}^{n-1} \tau s_k^2 + \sum_{k=1}^{n-1} \tau s_{\bar{t},k}^2 + \sum_{k=1}^{n-1} \tau s_{\bar{t}t,k}^2 + \frac{1}{3}\tau s_{\bar{t},1}^2 + \frac{1}{\tau} s_{\bar{t},1}^2 + C\tau$$

Now, for the remaining terms,

$$\tau s_{\bar{t},1}^2 \le \int_0^\tau \left|s'(t)\right|^2 \, dt, \quad \frac{1}{\tau} s_{\bar{t},1}^2 = \frac{1}{\tau^3} \left| \int_0^\tau \int_0^t s''(\xi)\, d\xi \, dt \right|^2 \le$$

$$\le \frac{1}{2\tau} \int_0^\tau \int_0^t \left|s''(\xi)\right|^2 \, d\xi \, dt \le \frac{1}{2} \int_0^\tau \left|s''(t)\right|^2 \, dt$$

Hence,

$$\|s\|^2_{W_2^2[0,T]} \le C_1, \ C_1 \text{ independent of } \tau.$$

By absolute continuity of the integral, it follows that

$$\lim_{\tau \to 0} \|s\|_{W_2^2[0,T]} = 0$$

Preliminary Results

$$\max\left(\|\|\|^2_{W_2^2[0,T]}; \|g\|^2_{W_2^1[0,T]}\right) \le \max\left(\|[s]_n\|^2_{w_2^2}; \|[g]_n\|^2_{w_2^1}\right) +$$
$$+C\tau + \frac{1}{2}\|s'\|^2_{W_2^1[0,\tau]} \le R^2 + C\tau + \frac{1}{2}\|s'\|^2_{W_2^1[0,\tau]}$$

For $\epsilon > 0$, choose n_ϵ such that

$$R^2 + C\tau + \frac{1}{2}\|s'\|^2_{W_2^1[0,\tau]} \le (R+\epsilon)^2$$

This gives equation (20), and the lemma is proved.

Corollary 8

Let either $[v]_n \in V_R^n$ or $[v]_n = \mathcal{Q}_n(v)$ for $v \in V_R$. Then

$$|s_k - s_{k-1}| \le C\tau, \quad k = 1, 2, \cdots, n \tag{22}$$

where C is independent of n.

The *Proof* depends on the previous lemma and Morrey's inequality:

1. If $v \in V_R$, then $s' \in W_2^1[0,T]$. Morrey's inequality applied to s' gives

$$\|s'\|_{C[0,T]} \leq C_1 \|s'\|_{W_2^1[0,T]} \leq C_1 R$$

Hence, for the first component of $[v]_n = Q_n(v)$,

$$[s]_n = \big(s(0), s(\tau), \dots, s(T)\big)$$

we have

$$|s_k - s_{k-1}| \leq \|s'\|_{C[0,T]} |t_k - t_{k-1}| \leq C\tau$$

2. If $[v]_n \in V_R^n$, then $v^n = P_n\big([v]_n\big) \in V_{R+1}$ for n large enough, and $\|(s^n)'\|_{C[0,T]} \leq C_1 R$ by the previous result.
Since

$$s^n(0) = s_0, \quad s^n(t_k) = \frac{s_k + s_{k-1}}{2}, \quad k = 1, \dots, n$$

we have again

$$|s_k - s_{k-1}| \leq C\tau$$

Lemma 9

(Tikhonov, Vasil'ev 1980) Sequence I_n approximates the continuous optimal control problem I if and only if the following conditions are satisfied:

1. *For arbitrary sufficiently small $\epsilon > 0$ there exists a number $N_1 = N_1(\epsilon)$ such that $\mathcal{Q}_N(v) \in V_R^n$ for all $v \in V_{R-\epsilon}$ and $N \geq N_1$; and for any fixed $\epsilon > 0$ and for all $v \in V_{R-\epsilon}$ the following inequality is satisfied:*
$$\limsup_{N \to \infty} \Big(I_N(\mathcal{Q}_N(v)) - J(v) \Big) \leq 0. \tag{23}$$
2. *For arbitrary sufficiently small $\epsilon > 0$ there exists a number $N_2 = N_2(\epsilon)$ such that $\mathcal{P}_N([v]_N) \in V_{R+\epsilon}$ for all $[v]_N \in V_R^N$ and $N \geq N_2$; and for all $[v]_N \in V_R^N$, $N \geq 1$ the following inequality is satisfied:*
$$\limsup_{N \to \infty} \Big(J(\mathcal{P}_N([v]_N)) - I_N([v]_N) \Big) \leq 0. \tag{24}$$
3. $\limsup_{\epsilon \to 0} J_*(\epsilon) \geq J_*, \quad \liminf_{\epsilon \to 0} J_*(-\epsilon) \leq J_*$, *where*
$$J_*(\pm\epsilon) = \inf_{V_{R\pm\epsilon}} J(u)$$

First Energy Estimate and its Consequences

Theorem 10

For all sufficiently small τ discrete state vector $[u([v]_n)]_n$ satisfies:

$$\max_{0\le k\le n}\int_0^l u^2(x;k)\,dx+\tau\sum_{k=1}^{n}\int_0^l\Big|\frac{du(x;k)}{dx}\Big|^2dx\le$$

$$C\Big(\|\phi\|^2_{L_2(0,s_0)}+\|g^n\|^2_{L_2(0,T)}+\|f\|^2_{L_2(D)}+\|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)}$$

$$+\|\chi(s^n(t),t)\|^2_{L_2(0,T)}+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}}u^2(x;k)dx\Big),\qquad(25)$$

$$\max_{0\le k\le n}\int_0^l u^2(x;k)\,dx+\tau\sum_{k=0}^{n}\int_0^l\Big|\frac{du(x;k)}{dx}\Big|^2dx+\tau^2\sum_{k=1}^{n}\int_0^l u^2_{\bar t}(x;k)dx\le$$

$$C\Big(\|\phi\|^2_{W_2^1(0,s_0)}+\|g^n\|^2_{L_2(0,T)}+\|f\|^2_{L_2(D)}+\|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)}$$

$$+\|\chi(s^n(t),t)\|^2_{L_2(0,T)}+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}}u^2(x;k)dx\Big),\qquad(26)$$

Lemma 11

For all sufficiently small τ, discrete state vector $[u([v]_n)]_n$ satisfies the following estimation:

$$\max_{1\le k\le n}\int_0^{s_k} u^2(x;k)\,dx+\tau\sum_{k=1}^{n}\int_0^{s_k}\Big|\frac{du(x;k)}{dx}\Big|^2dx+\tau^2\sum_{k=1}^{n}\int_0^{s_k}u_{\bar{t}}^2(x;k)dx\le$$

$$C\Big(\|\phi\|^2_{L_2(0,s_0)}+\|g^n\|^2_{L_2(0,T)}+\|f\|^2_{L_2(D)}+\|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)}$$

$$+\|\chi(s^n(t),t)\|^2_{L_2(0,T)}+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}}u^2(x;k)dx\Big),\quad (27)$$

where C is independent of τ.

Proof. Choose $\eta(x) = 2\tau u(x;k)$ in (15) and use the equality

$$2\tau u_{\bar{t}}(x;k)u(x;k) = u^2(x;k) - u^2(x;k-1) + \tau^2 u_{\bar{t}}^2(x;k)$$

$$\int_0^{s_k} u^2(x;k)dx - \int_0^{s_k} u^2(x;k-1)dx + \tau^2 \int_0^{s_k} u_{\bar{t}}^2(x;k)dx +$$
$$2\tau \int_0^{s_k} a_k(x)\Big|\frac{du(x;k)}{dx}\Big|^2 dx = 2\tau \int_0^{s_k} \Big[b_k(x)\frac{du(x;k)}{dx}u(x;k) + c_k(x)u^2(x;k) -$$
$$f_k(x)u(x;k)\Big]dx - 2\tau\left[(\gamma_{s^n}(s^n)')^k - \chi_{s^n}^k\right]u(s_k;k) - 2\tau g_k^n u(0;k). \quad (28)$$

$$\int_0^{s_k} u^2(x;k)dx - \int_0^{s_k} u^2(x;k-1)dx + a_0\tau \int_0^{s_k} \Big|\frac{du(x;k)}{dx}\Big|^2 dx +$$
$$\tau^2 \int_0^{s_k} u_{\bar{t}}^2(x;k)dx \le C_1\tau\Big[|(\gamma_{s^n}(s^n)')^k|^2 + |\chi_{s^n}^k|^2 + |g_k^n|^2 +$$
$$\int_0^{s_k} f_k^2(x)dx + \int_0^{s_k} u^2(x;k)dx\Big], \quad (29)$$

$$(1-C_1\tau)\int_0^{s_k} u^2(x;k)dx \le \int_0^{s_{k-1}} u^2(x;k-1)dx + \mathbf{1}_+(s_k - s_{k-1})\times$$
$$\int_{s_{k-1}}^{s_k} u^2(x;k-1)dx + C_1\tau\left[|(\gamma_{s^n}(s^n)')^k|^2 + |\chi^k_{s^n}|^2 + |g^n_k|^2 + \int_0^{s_k} f_k^2(x)dx\right], \tag{30}$$

By induction

$$\int_0^{s_k} u^2(x;k)dx \le (1-C_1\tau)^{-k}\int_0^{s_0}\phi^2(x)dx + \sum_{j=1}^{k}(1-C_1\tau)^{-k+j-1}\Big\{C_1\tau\times$$
$$\left[|(\gamma_{s^n}(s^n)')^j|^2 + |\chi^j_{s^n}|^2 + |g^n_j|^2 + \int_0^{s_j} f_j^2 dx\right]+$$
$$+\mathbf{1}_+(s_j - s_{j-1})\int_{s_{j-1}}^{s_j} u^2(x;j-1)dx\Big\}. \tag{31}$$

For arbitrary $1 \le j \le k \le n$ we have

$$(1-C_1\tau)^{-k+j-1} \le (1-C_1\tau)^{-k} \le (1-C_1\tau)^{-n} = \Big(1-\frac{C_1 T}{n}\Big)^{-n} \to e^{C_1 T}, \tag{32}$$

as $\tau \to 0$. Accordingly for sufficiently small τ we have

$$(1-C_1\tau)^{-k+j-1} \le 2e^{C_1 T} \quad \text{for } 1 \le j \le k \le n \tag{33}$$

By applying CBS inequality from (31)-(33) it follows that

$$\max_{1\le k\le n}\int_0^{s_k} u^2(x;k)\,dx \le C_2\Big(\|\phi\|^2_{L_2(0,s_0)}+\|g^n\|^2_{L_2(0,T)}+\|\chi(s^n(t),t)\|^2_{L_2(0,T)}+$$

$$\|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)}+\|f\|^2_{L_2(D)}+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}} u^2(x;k)dx\Big) \tag{34}$$

Perform summation of (29) with respect to k from 1 to n and derive

$$\int_0^{s_n} u^2(x;n)\,dx + a_0\tau\sum_{k=1}^{n}\int_0^{s_k}\Big|\frac{du(x;k)}{dx}\Big|^2 dx+\tau^2\sum_{k=1}^{n}\int_0^{s_k} u^2_{\bar{t}}(x;k)dx \le$$

$$2\|\phi\|^2_{L_2(0,s_0)}+C_1\Big(\|g^n\|^2_{L_2(0,T)}+\|f\|^2_{L_2(D)}+\|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)}$$

$$+\|\chi(s^n(t),t)\|^2_{L_2(0,T)}+\tau\sum_{k=1}^{n}\int_0^{s_k} u^2(x;k)dx\Big)+$$

$$\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}} u^2(x;k)dx, \tag{35}$$

From (34) and (35), (27) follows. Lemma is proved.

Extension Lemma

Lemma 12

Given discrete control vector $[v]_n \in V_R^n$, *a discrete state vector* $[u([v]_n)]_n$ *satisfies the inequality*

$$\max_{1\leq k\leq n} \int_0^l u^2(x;k)\,dx + \tau \sum_{k=0}^n \int_0^l \left|\frac{du(x;k)}{dx}\right|^2 dx + \tau \sum_{k=1}^n \int_0^l u_{\bar{t}}^2(x;k)dx \leq$$

$$C\Big(\max_{1\leq k\leq n} \int_0^{s_k} u^2(x;k)\,dx + \tau \sum_{k=0}^n \int_0^{s_k} \left|\frac{du(x;k)}{dx}\right|^2 dx + \tau \sum_{k=1}^n \int_0^{s_k} u_{\bar{t}}^2(x;k)dx\Big), \tag{36}$$

where C *is independent of* τ.

Proof. By induction it follows that the first two terms on the left hand side are estimated by the first two terms on the right hand side with the constant $C = 2^N$, where N is defined in (16).

$$\tilde{u}(y;0)=\phi(ys_0),\ \ \tilde{u}(y;k)=u(ys_k;k),\quad 0\le y\le 1, k=1,...,n.$$

$$\tilde{u}(y;k)=\tilde{u}(2^n-y;k),\quad \text{for } 2^{n-1}\le y\le 2^n.$$

$$\sum_{k=1}^{n}\tau\int\limits_0^l u_{\bar{t}}^2(x;k)dx\le\sum_{k=1}^{n}\tau\int\limits_0^{2^N s_k}u_{\bar{t}}^2(x;k)dx=$$

$$\sum_{k=1}^{n}\tau\int\limits_0^{2^N s_k}\Big[\frac{\tilde{u}(x/s_k;k)-\tilde{u}(x/s_{k-1};k-1)}{\tau}\Big]^2dx=$$

$$\sum_{k=1}^{n}\tau s_k\int\limits_0^{2^N}\Big[\frac{\tilde{u}(y;k)-\tilde{u}(ys_k/s_{k-1};k-1)}{\tau}\Big]^2dy\le I_1+I_2 \qquad (37)$$

$$I_1 = 2\sum_{k=1}^{n} \tau s_k \int\limits_0^{2^N} \Big[\frac{\tilde{u}(y;k)-\tilde{u}(y;k-1)}{\tau}\Big]^2 dy = \cdots = 2^{N+1}\sum_{k=1}^{n}\tau s_k \times$$

$$\int\limits_0^1 \tilde{u}^2_{\bar{t}}(y;k)dy = 2^{N+1}\sum_{k=1}^{n}\tau \int\limits_0^{s_k} \Big[\frac{u(x;k)-u(xs_{k-1}/s_k;k-1)}{\tau}\Big]^2 dx \leq 2^{N+2}\times$$

$$\sum_{k=1}^{n}\tau \int\limits_0^{s_k} u^2_{\bar{t}}(x;k)dx + 2^{N+2}\sum_{k=1}^{n}\tau \int\limits_0^{s_k} \Big[\frac{u(x;k-1)-u(xs_{k-1}/s_k;k-1)}{\tau}\Big]^2 dx \tag{38}$$

$$I_2 = 2\sum_{k=1}^{n} \tau s_k \int\limits_0^{2^N} \Big[\frac{\tilde{u}(y;k-1)-\tilde{u}(ys_k/s_{k-1};k-1)}{\tau}\Big]^2 dy.$$

$$\sum_{k=1}^{n}\tau\int_0^{s_k}\Big[\frac{u(x;k-1)-u(xs_{k-1}/s_k;k-1)}{\tau}\Big]^2dx=$$

$$\sum_{k=1}^{n}\frac{1}{\tau}\int_0^{s_k}\Big|\int_{x\frac{s_{k-1}}{s_k}}^{x}\frac{du(\xi;k-1)}{d\xi}d\xi\Big|^2dx\le\frac{C_1^2 l}{\delta}\sum_{k=0}^{n-1}\tau\int_0^{l}\Big|\frac{du(x;k)}{dx}\Big|^2dx, \tag{39}$$

$$I_2\le\frac{2^{2N+1}C_1^2N^2}{\delta}\sum_{k=1}^{n}\tau\int_0^{N2^N}\Big|\frac{d\tilde{u}(x;k-1)}{dx}\Big|^2dx=\frac{2^{3N+1}C_1^2N^3}{\delta}\times$$

$$\sum_{k=0}^{n-1}\tau\int_0^{1}\Big|\frac{d\tilde{u}(x;k)}{dx}\Big|^2dx\le\frac{2^{3N+1}C_1^2N^3l}{\delta}\sum_{k=0}^{n-1}\tau\int_0^{s_k}\Big|\frac{du(x;k)}{dx}\Big|^2dx \tag{40}$$

$$\sum_{k=1}^{n}\tau\int_0^{l}u_{\bar{t}}^2(x;k)dx\le C\Big(\sum_{k=0}^{n-1}\tau\int_0^{s_k}\Big|\frac{du(x;k)}{dx}\Big|^2dx+\sum_{k=1}^{n}\tau\int_0^{s_k}u_{\bar{t}}^2(x;k)dx\Big) \tag{41}$$

Weak Compactness in $V_2^{1,0}$

Theorem 13

Let $[v]_n \in V_R^n, n = 1, 2, ...$ *be a sequence of discrete controls and the sequence* $\{\mathcal{P}_n([v]_n\}$ *converges strongly in* $W_2^1[0,T] \times L_2[0,T]$ *to* $v = (s, g)$*. Then the sequence* $\{u^\tau\}$ *converges as* $\tau \to 0$ *weakly in* $W_2^{1,0}(\Omega)$ *to weak solution* $u \in V_2^{1,0}(\Omega)$ *of the problem (1)-(4), i.e. to the solution of the integral identity (14). Moreover,* u *satisfies the energy estimate*

$$\|u\|^2_{V_2^{1,0}(D)} \leq C\Big(\|\phi\|^2_{L_2(0,s_0)} + \|g\|^2_{L_2(0,T)} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,0}(D)} + \|\chi\|^2_{W_2^{1,0}(D)}\Big) \tag{42}$$

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