Topological Dynamics and Universality in Chaos
I. Proof of Sharkovski’s Theorem

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FIT Colloquium

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Let \( f : I \to I \) be a continuous map, and \( I \) be an interval. *Interval* is a connected subset of the real line which contains more than one point. \( < a, b > \) is a closed interval with endpoints \( a \) and \( b \).

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f^1 = f, \quad f^{n+1} = f \circ f^n, \quad n \geq 1
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Introduction

Let \( f : I \to I \) be a continuous map, and \( I \) be an interval. \textit{Interval} is a connected subset of the real line which contains more than one point. \( < a, b > \) is a closed interval with endpoints \( a \) and \( b \).

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Sharkovski’s Theorem

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\( \{ f^n(x) : n \geq 0 \} \) be an orbit of \( x \). \( c \in I \) is a fixed point if \( f(c) = c \).
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\[ \{ f^n(x) : n \geq 0 \} \] be an orbit of \( x \). \( c \in I \) is a fixed point if \( f(c) = c \). If \( I = [a, b] \) is compact \( \Rightarrow \exists \) a fixed point \( c \in I \), since

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f(a) - a \geq 0 \geq f(b) - b
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Sharkovski’s Theorem

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A point \(c \in I\) is said to be periodic point of \(f\) with period \(m\) if \(f^m(c) = c, f^k(c) \neq c\) for \(1 \leq k < m\).
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A point \( c \in I \) is said to be periodic point of \( f \) with period \( m \) if \( f^m(c) = c, f^k(c) \neq c \) for \( 1 \leq k < m \). Periodic \( m \)-orbit is
\[ c, f(c), f^2(c), \ldots, f^{m-1}(c) \]
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\{ f^n(x) : n \geq 0 \} be an orbit of x. \( c \in I \) is a fixed point if \( f(c) = c \). If 
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**Theorem 1**

*(Sharkovsky, 1964)* Let the positive integers be totally ordered in the 
following way:

\[ 1 \succ 2 \succ 2^2 \succ 2^3 \succ \ldots \succ 2^2 \cdot 5 \succ 2^2 \cdot 3 \succ \ldots \succ 2 \cdot 5 \succ 2 \cdot 3 \succ \ldots \succ 7 \succ 5 \succ 3 \]

If \( f \) has a cycle of period \( n \) and \( m \succ n \), then \( f \) also has a periodic orbit of period \( m \).
Proof of Sharkovski’s Theorem

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Lemma 2
If $J$ is a compact subinterval such that $J \subseteq f(J)$, then $f$ has a fixed point in $J$. 

Proof. If $J = [a, b]$ then for some $c, d \in J$ we have $f(c) = a$, $f(d) = b$. Thus $f(c) \leq c$, $f(d) \geq d$, and by the intermediate value theorem $\exists c^* \in [a, b] f(c^*) = c^*$. 

Lemma 3
If $J, K$ are compact subintervals such that $K \subseteq f(J)$, then there is a compact subinterval $L \subseteq J$ such that $f(L) = K$. 

Proof. Let $K = [a, b]$, $c = \sup\{x \in J : f(x) = a\}$. If $f(x) = b$ for some $x \in J$ with $x > c$, let $d$ be the least and take $L = [c, d]$. Otherwise $f(x) = b$ for some $x \in J$ with $x < c$. Let $c' = \text{greatest}$ and let $d' \leq c$ be the least $x \in J$ with $x > c'$ for which $f(x) = a$. Then we can take $L = [c', d']$. 

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Lemma 2

If $J$ is a compact subinterval such that $J \subseteq f(J)$, then $f$ has a fixed point in $J$.

Proof. If $J = [a, b]$ then for some $c, d \in J$ we have $f(c) = a$, $f(d) = b$. Thus $f(c) \leq c$, $f(d) \geq d$, and by the intermediate value theorem $\exists c_* \in [a, b] f(c_*) = c_*$. □
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Proof. Let $K = [a, b]$, $c = \sup \{x \in J : f(x) = a\}$. If $f(x) = b$ for some $x \in J$ with $x > c$, let $d$ be the least and take $L = [c, d]$. 
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Thus \( f(c) \leq c, \ f(d) \geq d \), and by the intermediate value theorem

\[ \exists c_* \in [a, b] \ f(c_*) = c_* \. \]

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Proof. Let \( K = [a, b], \ c = \sup\{x \in J : f(x) = a\} \). If \( f(x) = b \) for some \( x \in J \) with \( x > c \), let \( d \) be the least and take \( L = [c, d] \). Otherwise \( f(x) = b \) for some \( x \in J \) with \( x < c \). Let \( c' \) be the greatest and let \( d' \leq c \) be the least \( x \in J \) with \( x > c' \) for which \( f(x) = a \). Then we can take \( L = [c', d'] \).
Lemma 4

If $J_0, J_1, ..., J_m$ are compact subintervals such that $J_k \subseteq f(J_{k-1})$ ($1 \leq k \leq m$), then there is a compact subinterval $L \subseteq J_0$ such that $f^m(L) = J_m$ and $f^k(L) \subseteq J_k$ ($1 \leq k < m$).
**Lemma 4**

If $J_0, J_1, ..., J_m$ are compact subintervals such that $J_k \subseteq f(J_{k-1})$ $(1 \leq k \leq m)$, then there is a compact subinterval $L \subseteq J_0$ such that $f^m(L) = J_m$ and $f^k(L) \subseteq J_k$ $(1 \leq k < m)$.

If also $J_0 \subseteq J_m$, then there exists a point $y$ such that $f^m(y) = y$ and $f^k(y) \in J_k$ $(0 \leq k < m)$. 
Lemma 4

If \( J_0, J_1, ..., J_m \) are compact subintervals such that \( J_k \subseteq f(J_{k-1}) \) 
(1 \( \leq k \leq m \)), then there is a compact subinterval \( L \subseteq J_0 \) such that 
\( f^m(L) = J_m \) and \( f^k(L) \subseteq J_k \) (1 \( \leq k < m \)).

If also \( J_0 \subseteq J_m \), then there exists a point \( y \) such that 
\( f^m(y) = y \) and \( f^k(y) \in J_k \) (0 \( \leq k < m \)).

Proof. Note that the first assertion holds for \( m = 1 \) due to Lemma 3:

\[
J_1 \subseteq f(J_0) \Rightarrow \exists L \subseteq J_0, f(L) = J_1
\]

Prove by induction: let \( m > 1 \) be fixed and assume the assertion is true 
for all smaller values. We have

\[
J_1 \subseteq f(J_0), J_2 \subseteq f(J_1), ..., J_{m-1} \subseteq f(J_{m-2}), J_m \subseteq f(J_{m-1})
\]

Induction assumption applied to last \( m - 1 \) relations \( \Rightarrow \)

\[
\exists L' \subseteq J_1 : f^{m-1}(L') = J_m, f^k(L') \subseteq J_{k+1}, 1 \leq k < m - 1
\]
Now we have $L' \subseteq J_1 \subseteq f(J_0) \Rightarrow \exists L \subseteq J_0$ such that $f(L) = L' \subseteq J_1 \Rightarrow$

$$f^m(L) = f^{m-1}(L') = J_m; \quad f^k(L) = f^{k-1}(L') \subseteq J_k, \quad 2 \leq k < m$$

Hence first assertion of the lemma is proved.
Now we have $L' \subseteq J_1 \subseteq f(J_0) \Rightarrow \exists L \subseteq J_0$ such that $f(L) = L' \subseteq J_1 \Rightarrow$

$$f^m(L) = f^{m-1}(L') = J_m; \ f^k(L) = f^{k-1}(L') \subseteq J_k, 2 \leq k < m$$

Hence first assertion of the lemma is proved.

If also $J_0 \subseteq J_m$, from the first assertion $\Rightarrow$

$$\exists L \subseteq J_0 : \ f^m(L) = J_m \supseteq J_0 \supseteq L$$

and by Lemma 2 we have

$$\exists c \in L \subseteq J_0 : \ f^m(c) = c, \ \text{and} \ f^k(c) \subseteq J_k, 1 \leq k < m$$

Lemma 4 is proved.
Lemma 5

*Between any two points of a periodic orbit of period $n > 1$ there is a point of a periodic orbit of period less than $n$.*

Proof. Let $a < b$ are two adjacent points of $n$-orbit. Consider all $m < n$ such that $f^m(b) < b$. There is at least one such $m$. Since there is one more point of the orbit to the left of $b$ than to the left of $a$, for some $m$ such that $1 \leq m < n$ we have $f^m(a) > a, f^m(b) < b$. If $f^m$ is defined on $[a, b]$, then $\exists c \in (a, b) \ f^m(c) = c.$
Lemma 5

Between any two points of a periodic orbit of period $n > 1$ there is a point of a periodic orbit of period less than $n$.

Proof. Let $a < b$ are two adjacent points of $n$-orbit. Consider all $m < n$ such that $f^m(b) < b$. There is at least one such $m$. Since there is one more point of the orbit to the left of $b$ than to the left of $a$, for some $m$ such that $1 \leq m < n$ we have $f^m(a) > a$, $f^m(b) < b$. If $f^m$ is defined on $[a, b]$, then $\exists c \in (a, b)$ $f^m(c) = c$.

Assume $f^m$ is not defined throughout $[a, b]$. Let $J_k = < f^k(a), f^k(b) >$, $1 \leq k \leq m$. We have

$$J_k \subseteq f(J_{k-1}), 1 \leq k \leq m$$

We also have $J_0 \subseteq J_m$, since $f^m(a) \geq b$, $f^m(b) \leq a$. hence, the assertion of the Lemma follows from Lemma 4.
Let $B = \{x_1 < x_2 < \cdots < x_n\}$ be $n$-orbit of $f$.

**Definition 1.1**

If $f(x_i) = x_{s_i}$, $1 \leq s_i \leq n$, $i = 1, 2, \ldots, n$, then $B$ is associated with cyclic permutation

$$
\begin{bmatrix}
1 & 2 & \cdots & n \\
\text{s} & \text{s} & \cdots & \text{s}
\end{bmatrix}
$$

**Definition 1.2**

Let $I_i = [x_i, x_{i+1}]$. Digraph of a cycle is a directed graph of transitions with vertices $I_1, I_2, \ldots, I_{n-1}$ and oriented edges $I_i \rightarrow I_s$ if $I_s \subseteq f(I_i)$. 

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Proof of Sharkovski’s Theorem

Properties of Digraphs:

1. \( \forall I_j \exists \text{ at least one } I_k \text{ for which } I_j \rightarrow I_k \). Moreover, it is always possible to choose \( k \neq j \), unless \( n = 2 \).
Proof of Sharkovski’s Theorem

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Proof: Suppose there is no \( j \neq k \) such that \( I_j \rightarrow I_k \). Then if \( i \neq k \),
\[
f(x_i) \leq x_k \Rightarrow f(x_{i+1}) \leq x_k \text{ and } f(x_i) \geq x_{k+1} \Rightarrow f(x_{i+1}) \geq x_{k+1}.
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\]
If \( f(x_{k+1}) \geq x_{k+1} \Rightarrow f(x_i) \geq x_{k+1} \) for \( k < i \leq n \Rightarrow \) proper subset \( \{x_{k+1}, \ldots, x_n\} \) is mapped to itself. Hence, \( f(x_{k+1}) \leq x_k \), and similarly \( f(x_k) \geq x_{k+1} \), and accordingly there is a loop \( I_k \to I_k \).
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Properties of Digraphs:

1. ∀ I_j ∃ at least one I_k for which I_j → I_k. Moreover, it is always possible to choose k ≠ j, unless n = 2.

2. ∀ I_k ∃ at least one I_j for which I_j → I_k. Moreover, it is always possible to choose j ≠ k, unless n is even and k = \frac{n}{2}.

Proof: Suppose there is no j ≠ k such that I_j → I_k. Then if i ≠ k, f(x_i) ≤ x_k ⇒ f(x_{i+1}) ≤ x_k and f(x_i) ≥ x_{k+1} ⇒ f(x_{i+1}) ≥ x_{k+1}. If f(x_{k+1}) ≥ x_{k+1} ⇒ f(x_i) ≥ x_{k+1} for k < i ≤ n ⇒ proper subset \{x_{k+1}, ..., x_n\} is mapped to itself. Hence, f(x_{k+1}) ≤ x_k, and similarly f(x_k) ≥ x_{k+1}, and accordingly there is a loop I_k → I_k.

\[
f(x_i) ≤ x_k, k < i ≤ n ⇒ n - k ≤ k ⇒ n ≤ 2k,
\]

\[
f(x_i) ≥ x_{k+1}, 1 ≤ i ≤ k ⇒ k ≤ n - k ⇒ n ≥ 2k ⇒ n = 2k \qed
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If \( f(x_{k+1}) \geq x_{k+1} \Rightarrow f(x_i) \geq x_{k+1} \) for \( k < i \leq n \) \( \Rightarrow \) proper subset \( \{x_{k+1}, \ldots, x_n\} \) is mapped to itself. Hence, \( f(x_{k+1}) \leq x_k \), and similarly \( f(x_k) \geq x_{k+1} \), and accordingly there is a loop \( I_k \to I_k \).

\[
f(x_i) \leq x_k, k < i \leq n \Rightarrow n - k \leq k \Rightarrow n \leq 2k,
\]

\[
f(x_i) \geq x_{k+1}, 1 \leq i \leq k \Rightarrow k \leq n - k \Rightarrow n \geq 2k \Rightarrow n = 2k \quad \square
\]

3. Digraph always contains a loop

We have \( f(x_1) > x_1, f(x_n) < x_n \). Let

\[
k = \min\{1 \leq j < n : f(x_j) \geq x_{j+1}, f(x_{j+1}) \leq x_j\}
\]

Then \( I_k \to I_k \).
Definition 1.3

Given an n-orbit, a cycle

\[ J_0 \rightarrow J_1 \rightarrow \cdots \rightarrow J_{n-1} \rightarrow J_0 \]

of length n in the digraph is called a Fundamental Cycle (FC) if

\( J_0 \) contains an endpoint \( c \) s.t. \( f^k(c) \) is an endpoint of \( J_k \) for \( 1 \leq k < n \).
Definition 1.3
Given $n$-orbit, a cycle

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of length $n$ in the digraph is called a **Fundamental Cycle** (FC) if $J_0$ contains an endpoint $c$ s.t. $f^k(c)$ is an endpoint of $J_k$ for $1 \leq k < n$. FC always exists and unique.
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Given an $n$-orbit, a cycle

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of length $n$ in the digraph is called a **Fundamental Cycle (FC)** if $J_0$ contains an endpoint $c$ s.t. $f^k(c)$ is an endpoint of $J_k$ for $1 \leq k < n$.

FC always exists and unique. In the FC some vertex must occur at least twice among $J_0, \ldots, J_{n-1}$, since digraph has only $n - 1$ vertices.
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FC always exists and unique. In the FC some vertex must occur at least twice among \( J_0, ..., J_{n-1} \), since digraph has only \( n - 1 \) vertices. On the other hand, every vertex occurs at most twice, since interval \( I_k \) has two endpoints.
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Definition 1.4

*Cycle in a digraph is said to be primitive if it does not consist entirely of a cycle of smaller length described several times.*
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of length \( n \) in the digraph is called a **Fundamental Cycle** (FC) if \( J_0 \) contains an endpoint \( c \) s.t. \( f^k(c) \) is an endpoint of \( J_k \) for \( 1 \leq k < n \).

FC always exists and unique. In the FC some vertex must occur at least twice among \( J_0, \ldots, J_{n-1} \), since digraph has only \( n - 1 \) vertices. On the other hand, every vertex occurs at most twice, since interval \( I_k \) has two endpoints.

Definition 1.4
Cycle in a digraph is said to be **primitive** if it does not consist entirely of a cycle of smaller length described several times.

If FC contains \( I_k \) twice then it can be decomposed into two cycles of smaller length, each of which contains \( I_k \) once, and consequently is primitive.
Lemma 6

Suppose \( f \) has a periodic point of period \( n > 1 \). If the associated digraph contains a primitive cycle

\[ J_0 \rightarrow J_1 \rightarrow \cdots \rightarrow J_{m-1} \rightarrow J_0 \]

of length \( m \), then \( f \) has a periodic point \( y \) of period \( m \) such that \( f^k(y) \in J_k (0 \leq k < m) \).

Proof. \( J_1 \subseteq f(J_0), J_2 \subseteq f(J_1), \cdots, J_{m-1} \subseteq f(J_{m-2}), J_0 \subseteq f(J_{m-1}) \)
Lemma 6

Suppose $f$ has a periodic point of period $n > 1$. If the associated digraph contains a primitive cycle

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of length $m$, then $f$ has a periodic point $y$ of period $m$ such that $f^k(y) \in J_k (0 \leq k < m)$.

Proof. $J_1 \subseteq f(J_0), J_2 \subseteq f(J_1), \ldots, J_{m-1} \subseteq f(J_{m-2}), J_0 \subseteq f(J_{m-1})$

Lemma 4 (w. $J_m = J_0$) $\Rightarrow \exists y \in J_0 f^m(y) = y, f^k(y) \in J_k (0 \leq k < m)$
Lemma 6

Suppose $f$ has a periodic point of period $n > 1$. If the associated digraph contains a primitive cycle

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of length $m$, then $f$ has a periodic point $y$ of period $m$ such that $f^k(y) \in J_k (0 \leq k < m)$.

Proof. $J_1 \subseteq f(J_0)$, $J_2 \subseteq f(J_1)$, $\cdots$, $J_{m-1} \subseteq f(J_{m-2})$, $J_0 \subseteq f(J_{m-1})$

Lemma 4 (w. $J_m = J_0$) $\Rightarrow \exists y \in J_0$ $f^m(y) = y$, $f^k(y) \in J_k (0 \leq k < m)$

Either $m$ is a period of $y$ or period of $y$ is a factor of $m$. 
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Suppose $f$ has a periodic point of period $n > 1$. If the associated digraph contains a primitive cycle

$$J_0 \to J_1 \to \cdots \to J_{m-1} \to J_0$$

of length $m$, then $f$ has a periodic point $y$ of period $m$ such that $f^k(y) \in J_k (0 \leq k < m)$.

Proof. $J_1 \subseteq f(J_0), J_2 \subseteq f(J_1), \cdots , J_{m-1} \subseteq f(J_{m-2}), J_0 \subseteq f(J_{m-1})$

Lemma 4 (w. $J_m = J_0$) $\Rightarrow \exists y \in J_0$ $f^m(y) = y, f^k(y) \in J_k (0 \leq k < m)$

Either $m$ is a period of $y$ or period of $y$ is a factor of $m$. If $y$ is not an endpoint of $J_0$, then $m$ is a period of $y$ since cycle is primitive.
Lemma 6
Suppose $f$ has a periodic point of period $n > 1$. If the associated digraph contains a primitive cycle

$$J_0 \to J_1 \to \cdots \to J_{m-1} \to J_0$$

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Either $m$ is a period of $y$ or period of $y$ is a factor of $m$. If $y$ is not an endpoint of $J_0$, then $m$ is a period of $y$ since cycle is primitive. Assume $y$ is an endpoint of $J_0$. Since $y$ is an element of $n$-orbit $\Rightarrow n$ is a divisor of $m$. We have $J_k \subseteq f(J_{k-1})$ and $f^k(y) \in J_k \Rightarrow J_k$ is defined uniquely and moreover, cycle is a multiple of the FC. Contradiction, since cycle is primitive.
Proof of Sharkovski’s Theorem

Straffin’s Lemma $\Rightarrow 3 \prec m \prec 2 \prec 1$

Suppose $f$ has a 3-orbit: $f(c) < c < f^2(c)$ with corresponding digraph

$\circlearrowleft I_1 \iff I_2$
Proof of Sharkovski’s Theorem

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Proof of Sharkovski’s Theorem

Straffin’s Lemma \( \Rightarrow 3 < m < 2 < 1 \)

Suppose \( f \) has a 3-orbit: \( f(c) < c < f^2(c) \) with corresponding digraph

\[
\bigcirc I_1 \Leftrightarrow I_2
\]

\( I_1 \rightarrow I_1 \Rightarrow \) there is a fixed point; \( I_1 \rightarrow I_2 \rightarrow I_1 \Rightarrow \) there is a 2-orbit

\( \forall \) positive integer \( m > 2 \) there is an \( m \)-orbit corresponding to primitive cycle of length \( m: I_1 \rightarrow I_2 \rightarrow I_1 \rightarrow I_1 \rightarrow \cdots \rightarrow I_1 \)

Lemma 7

If \( f \) has a periodic point of period \( > 1 \), then it has a fixed point and a periodic point of period 2.

Proof. Digraph has a loop \( \Rightarrow \) there is a fixed point. Let \( n > 1 \) be the least positive integer such that \( f \) has a periodic point of period \( n \). If \( n > 2 \) decompose FC into two primitive cycles. Since at least one of these has length greater than 1, by Straffin’s lemma we deduce there is a periodic point of period strictly between 1 and \( n \).
Lemma 8
Suppose \( f \) has a periodic orbit of odd period \( n > 1 \), but no periodic orbit of odd period strictly between 1 and \( n \). If \( c \) is the midpoint of the orbit of odd period \( n \), then the points of this orbit have the order

\[
    f^{n-1}(c) < f^{n-3}(c) < \ldots < f^2(c) < c < f(c) < \ldots < f^{n-2}(c)
\]

or the inverse order

\[
    f^{n-2}(c) < \ldots < f(c) < c < f^2(c) < \ldots < f^{n-3}(c) < f^{n-1}(c)
\]

and associated digraph is given in the figure, where \( J_1 = \langle c, f(c) \rangle \) and \( J_k = \langle f^{k-2}(c), f^k(c) \rangle \) for \( 1 < k < n \).
Proof of Sharkovski’s Theorem

Proof of Lemma 8

Proof. Decompose FC into two primitive cycles, one of which has odd length.
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where \( J_i \neq J_1 \) for \( 1 < i < n \). If \( J_i = J_k, \ 1 < i < k < n \), then we obtain a smaller primitive cycle, and by excluding the loop at \( J_1 \) if necessary, we can arrange that its length is odd. Hence, \( J_1, \ldots, J_{n-1} \) are all distinct and thus a permutation of \( I_1, \ldots, I_{n-1} \).
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Proof of Sharkovski’s Theorem

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\[ x_k = a, \ x_{k+1} = f(a), \ x_{k-1} = f^2(a) \]

or

\[ x_{k+1} = b, \ x_k = f(b), \ x_{k+2} = f^2(b). \]
Proof of Lemma 8

Consider the first case, the argument in the second being similar. If $f^3(a) < f^2(a)$ then $J_2 \rightarrow J_1$, which is forbidden. Hence $f^3(a) > f^2(a)$. 
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\( f^3(a) < f^2(a) \) then \( J_2 \to J_1 \), which is forbidden. Hence \( f^3(a) > f^2(a) \). Since \( J_2 \) is not directed to \( J_k \) for \( k > 3 \) it follows that \( J_3 = [f(a), f^3(a)] \) is adjacent to \( J_1 \) on the right.
Consider the first case, the argument in the second being similar. If $f^3(a) < f^2(a)$ then $J_2 \to J_1$, which is forbidden. Hence $f^3(a) > f^2(a)$. Since $J_2$ is not directed to $J_k$ for $k > 3$ it follows that $J_3 = [f(a), f^3(a)]$ is adjacent to $J_1$ on the right. If $f^4(a) > f^3(a)$ then $J_3 \to J_1$, which is forbidden. Hence $f^4(a) < f^2(a)$ and, since $J_3$ is not directed to $J_k$ for $k > 4$, $J_4 = [f^4(a), f^2(a)]$ is adjacent to $J_2$ on the left.
Consider the first case, the argument in the second being similar. If \( f^3(a) < f^2(a) \) then \( J_2 \rightarrow J_1 \), which is forbidden. Hence \( f^3(a) > f^2(a) \). Since \( J_2 \) is not directed to \( J_k \) for \( k > 3 \) it follows that \( J_3 = [f(a), f^3(a)] \) is adjacent to \( J_1 \) on the right. If \( f^4(a) > f^3(a) \) then \( J_3 \rightarrow J_1 \), which is forbidden. Hence \( f^4(a) < f^2(a) \) and, since \( J_3 \) is not directed to \( J_k \) for \( k > 4 \), \( J_4 = [f^4(a), f^2(a)] \) is adjacent to \( J_2 \) on the left. Proceeding in this way we see that the order of the \( J_i \)s on the real line is given by

\[
J_{n-1} \quad \cdots \quad J_4 \quad J_2 \quad J_1 \quad J_3 \quad \cdots \quad J_{n-2}
\]

\[
f^{n-1}(a) \quad f^{n-3}(a) \quad f^4(a) \quad f^2(a) \quad a \quad f(a) \quad f^3(a) \quad \cdots \quad f^{n-4}(a) \quad f^{n-2}(a)
\]

Since the endpoints of \( J_{n-1} = [x_1, x_2] \) are mapped into \( a \) and \( f^{n-2}(a) = x_n \) we have \( J_{n-1} \rightarrow J_k \) iff \( k \) is odd. We found all the arcs in the digraph. \( \square \)