Topological Dynamics and Universality in Chaos
I. Proof of Sharkovski’s Theorem

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Let $f : I \rightarrow I$ be a continuous map, and $I$ be an interval. *Interval* is a connected subset of the real line which contains more than one point. $\langle a, b \rangle$ is a closed interval with endpoints $a$ and $b$.

\[ f^1 = f, \quad f^{n+1} = f \circ f^n, \quad n \geq 1 \]
Sharkovski’s Theorem

\{f^n(x) : n \geq 0\} be an orbit of \(x\). \(c \in I\) is a fixed point if \(f(c) = c\). If \(I = [a, b]\) is compact \(\Rightarrow \exists\) a fixed point \(c \in I\), since

\[ f(a) - a \geq 0 \geq f(b) - b \]

A point \(c \in I\) is said to be periodic point of \(f\) with period \(m\) if \(f^m(c) = c, f^k(c) \neq c\) for \(1 \leq k < m\). Periodic \(m\)-orbit is

\[ c, f(c), f^2(c), \ldots, f^{m-1}(c) \]

**Theorem 1**

*(Sharkovsky, 1964)* Let the positive integers be totally ordered in the following way:

\[ 1 \succ 2 \succ 2^2 \succ 2^3 \succ \ldots \succ 2^2 \cdot 5 \succ 2^2 \cdot 3 \succ \ldots \succ 2 \cdot 5 \succ 2 \cdot 3 \succ \ldots \succ 7 \succ 5 \succ 3 \]

If \(f\) has a cycle of period \(n\) and \(m \succ n\), then \(f\) also has a periodic orbit of period \(m\).
Proof of Sharkovski’s Theorem

References


Lemma 2
If $J$ is a compact subinterval such that $J \subseteq f(J)$, then $f$ has a fixed point in $J$.

Proof. If $J = [a, b]$ then for some $c, d \in J$ we have $f(c) = a$, $f(d) = b$. Thus $f(c) \leq c$, $f(d) \geq d$, and by the intermediate value theorem, there exists $c_* \in [a, b]$ such that $f(c_*) = c_*$. □

Lemma 3
If $J, K$ are compact subintervals such that $K \subseteq f(J)$, then there is a compact subinterval $L \subseteq J$ such that $f(L) = K$.

Proof. Let $K = [a, b], c = \sup\{x \in J : f(x) = a\}$. If $f(x) = b$ for some $x \in J$ with $x > c$, let $d$ be the least and take $L = [c, d]$. Otherwise, $f(x) = b$ for some $x \in J$ with $x < c$. Let $c'$ be the greatest and let $d' \leq c$ be the least $x \in J$ with $x > c'$ for which $f(x) = a$. Then we can take $L = [c', d']$. □
Proof of Sharkovski’s Theorem

**Lemma 4**

If $J_0, J_1, ..., J_m$ are compact subintervals such that $J_k \subseteq f(J_{k-1})$ $(1 \leq k \leq m)$, then there is a compact subinterval $L \subseteq J_0$ such that $f^m(L) = J_m$ and $f^k(L) \subseteq J_k$ $(1 \leq k < m)$. If also $J_0 \subseteq J_m$, then there exists a point $y$ such that $f^m(y) = y$ and $f^k(y) \in J_k$ $(0 \leq k < m)$.

Proof. Note that the first assertion holds for $m = 1$ due to Lemma 3:

$$J_1 \subseteq f(J_0) \Rightarrow \exists L \subseteq J_0, f(L) = J_1$$

Prove by induction: let $m > 1$ be fixed and assume the assertion is true for all smaller values. We have

$$J_1 \subseteq f(J_0), J_2 \subseteq f(J_1), ..., J_{m-1} \subseteq f(J_{m-2}), J_m \subseteq f(J_{m-1})$$

Induction assumption applied to last $m - 1$ relations $\Rightarrow$

$$\exists L' \subseteq J_1 : f^{m-1}(L') = J_m, f^k(L') \subseteq J_{k+1}, 1 \leq k < m - 1$$
Proof of Sharkovski’s Theorem

Now we have $L' \subseteq J_1 \subseteq f(J_0) \Rightarrow \exists L \subseteq J_0$ such that $f(L) = L' \subseteq J_1 \Rightarrow$

$$f^m(L) = f^{m-1}(L') = J_m; \quad f^k(L) = f^{k-1}(L') \subseteq J_k, \quad 2 \leq k < m$$

Hence first assertion of the lemma is proved.

If also $J_0 \subseteq J_m$, from the first assertion $\Rightarrow$

$$\exists L \subseteq J_0: \quad f^m(L) = J_m \supseteq J_0 \supseteq L$$

and by Lemma 2 we have

$$\exists c \in L \subseteq J_0: \quad f^m(c) = c, \quad \text{and} \quad f^k(c) \subseteq J_k, \quad 1 \leq k < m$$

Lemma 4 is proved.
Lemma 5

*Between any two points of a periodic orbit of period \( n > 1 \) there is a point of a periodic orbit of period less than \( n \).*

Proof. Let \( a < b \) are two adjacent points of \( n \)-orbit. Consider all \( m < n \) such that \( f^m(b) < b \). There is at least one such \( m \). Since there is one more point of the orbit to the left of \( b \) than to the left of \( a \), for some \( m \) such that \( 1 \leq m < n \) we have \( f^m(a) > a, f^m(b) < b \). If \( f^m \) is defined on \([a, b]\), then \( \exists c \in (a, b) \ f^m(c) = c \).

Assume \( f^m \) is not defined throughout \([a, b]\). Let \( J_k = \langle f^k(a), f^k(b) \rangle \), \( 1 \leq k \leq m \). We have

\[
J_k \subseteq f(J_{k-1}), \ 1 \leq k \leq m
\]

We also have \( J_0 \subseteq J_m \), since \( f^m(a) \geq b, f^m(b) \leq a \). hence, the assertion of the Lemma follows from Lemma 4.
Proof of Sharkovski’s Theorem

Let \( B = \{x_1 < x_2 < \cdots < x_n\} \) be \( n \)-orbit of \( f \).

**Definition 1.1**

If \( f(x_i) = x_{s_i}, 1 \leq s_i \leq n, i = 1, 2, \ldots, n \), then \( B \) is associated with cyclic permutation\
\[
\begin{bmatrix}
1 & 2 & \cdots & n \\
 s_1 & s_2 & \cdots & s_n \\
\end{bmatrix}
\]

**Definition 1.2**

Let \( I_i = [x_i, x_{i+1}] \). Digraph of a cycle is a directed graph of transitions with vertices \( I_1, I_2, \ldots, I_{n-1} \) and oriented edges \( I_i \to I_s \) if \( I_s \subseteq f(I_i) \).
Proof of Sharkovski’s Theorem

Properties of Digraphs:

1. \( \forall I_j \ni \text{at least one } I_k \text{ for which } I_j \rightarrow I_k. \) Moreover, it is always possible to choose \( k \neq j \), unless \( n = 2 \).

2. \( \forall I_k \ni \text{at least one } I_j \text{ for which } I_j \rightarrow I_k. \) Moreover, it is always possible to choose \( j \neq k \), unless \( n \) is even and \( k = \frac{n}{2} \).

Proof: Suppose there is no \( j \neq k \) such that \( I_j \rightarrow I_k \). Then if \( i \neq k \),
\[
 f(x_i) \leq x_k \implies f(x_{i+1}) \leq x_k \quad \text{and} \quad f(x_i) \geq x_{k+1} \implies f(x_{i+1}) \geq x_{k+1}.
\]
If \( f(x_{k+1}) \geq x_{k+1} \implies f(x_i) \geq x_{k+1} \) for \( k < i \leq n \Rightarrow \text{proper subset } \{x_{k+1}, \ldots, x_n\} \text{ is mapped to itself. Hence, } f(x_{k+1}) \leq x_k, \text{ and similarly } f(x_k) \geq x_{k+1}, \text{ and accordingly there is a loop } I_k \rightarrow I_k.
\]
\[
 f(x_i) \leq x_k, k < i \leq n \Rightarrow n - k \leq k \Rightarrow n \leq 2k,
\]
\[
 f(x_i) \geq x_{k+1}, 1 \leq i \leq k \Rightarrow k \leq n - k \Rightarrow n \geq 2k \Rightarrow n = 2k \hspace{1cm} \square
\]

3. Digraph always contains a loop

We have \( f(x_1) > x_1, f(x_n) < x_n \). Let
\[
 k = \min\{1 \leq j < n : f(x_j) \geq x_{j+1}, f(x_{j+1}) \leq x_j\}
\]
Then \( I_k \rightarrow I_k \).
Proof of Sharkovski’s Theorem

Fundamental Cycle

**Definition 1.3**

*Given* \( n \)-*orbit, a cycle*

\[
J_0 \rightarrow J_1 \rightarrow \cdots \rightarrow J_{n-1} \rightarrow J_0
\]

*of length* \( n \) *in the digraph is called a Fundamental Cycle (FC) if* \( J_0 \)

*contains an endpoint* \( c \) *s.t. \( f^k(c) \) is an endpoint of* \( J_k \) *for* \( 1 \leq k < n \).*

FC always exists and unique. In the FC some vertex must occur at least twice among \( J_0, \ldots, J_{n-1} \), since digraph has only \( n - 1 \) vertices. On the other hand, every vertex occurs at most twice, since interval \( I_k \) has two endpoints.

**Definition 1.4**

*Cycle in a digraph is said to be primitive if it does not consist entirely of a cycle of smaller length described several times.*

If FC contains \( I_k \) twice then it can be decomposed into two cycles of smaller length, each of which contains \( I_k \) once, and consequently is primitive.
**Lemma 6**

Suppose \( f \) has a periodic point of period \( n > 1 \). If the associated digraph contains a primitive cycle

\[
J_0 \rightarrow J_1 \rightarrow \cdots \rightarrow J_{m-1} \rightarrow J_0
\]

of length \( m \), then \( f \) has a periodic point \( y \) of period \( m \) such that \( f^k(y) \in J_k (0 \leq k < m) \).

Proof. \( J_1 \subseteq f(J_0), J_2 \subseteq f(J_1), \cdots, J_{m-1} \subseteq f(J_{m-2}), J_0 \subseteq f(J_{m-1}) \)

Lemma 4 (w. \( J_m = J_0 \)) \( \Rightarrow \exists \ y \in J_0 \ f^m(y) = y, \ f^k(y) \in J_k (0 \leq k < m) \)

Either \( m \) is a period of \( y \) or period of \( y \) is a factor of \( m \). If \( y \) is not an endpoint of \( J_0 \), then \( m \) is a period of \( y \) since cycle is primitive. Assume \( y \) is an endpoint of \( J_0 \). Since \( y \) is an element of \( n \)-orbit \( \Rightarrow n \) is a divisor of \( m \). We have \( J_k \subseteq f(J_{k-1}) \) and \( f^k(y) \in J_k \) \( \Rightarrow J_k \) is defined uniquely and moreover, cycle is a multiple of the FC. Contradiction, since cycle is primitive.
Proof of Sharkovski’s Theorem

Straffin’s Lemma \( \Rightarrow \) \( 3 < m < 2 < 1 \)

Suppose \( f \) has a 3-orbit: \( f(c) < c < f^2(c) \) with corresponding digraph

\[
\circ I_1 \leftrightarrow I_2
\]

\( I_1 \rightarrow I_1 \Rightarrow \) there is a fixed point; \( I_1 \rightarrow I_2 \rightarrow I_1 \Rightarrow \) there is a 2-orbit

\( \forall \) positive integer \( m > 2 \) there is an \( m \)-orbit corresponding to primitive cycle of length \( m \): \( I_1 \rightarrow I_2 \rightarrow I_1 \rightarrow I_1 \rightarrow \cdots \rightarrow I_1 \)

**Lemma 7**

*If \( f \) has a periodic point of period \( > 1 \), then it has a fixed point and a periodic point of period 2.*

Proof. Digraph has a loop \( \Rightarrow \) there is a fixed point. Let \( n > 1 \) be the least positive integer such that \( f \) has a periodic point of period \( n \). If \( n > 2 \) decompose FC into two primitive cycles. Since at least one of these has length greater than 1, by Straffin’s lemma we deduce there is a periodic point of period strictly between 1 and \( n \).
Lemma 8
Suppose \( f \) has a periodic orbit of odd period \( n > 1 \), but no periodic orbit of odd period strictly between 1 and \( n \). If \( c \) is the midpoint of the orbit of odd period \( n \), then the points of this orbit have the order

\[
f^{n-1}(c) < f^{n-3}(c) < \cdots < f^2(c) < c < f(c) < \cdots < f^{n-2}(c)
\]
or the inverse order

\[
f^{n-2}(c) < \cdots < f(c) < c < f^2(c) < \cdots < f^{n-3}(c) < f^{n-1}(c)
\]

and associated digraph is given in the figure, where \( J_1 = < c, f(c) > \) and \( J_k = < f^{k-2}(c), f^k(c) > \) for \( 1 < k < n \).
Proof. Decompose FC into two primitive cycles, one of which has odd length. Its length must be 1! Thus FC has the form

\[ J_1 \rightarrow J_1 \rightarrow J_2 \rightarrow \cdots \rightarrow J_{n-1} \rightarrow J_1 \]

where \( J_i \neq J_1 \) for \( 1 < i < n \). If \( J_i = J_k, 1 < i < k < n \), then we obtain a smaller primitive cycle, and by excluding the loop at \( J_1 \) if necessary, we can arrange that its length is odd. Hence, \( J_1, \ldots, J_{n-1} \) are all distinct and thus a permutation of \( I_1, \ldots, I_{n-1} \). Similarly, we cannot have \( J_i \rightarrow J_k \) if \( k > i + 1 \) or if \( k = 1 \) and \( i \neq 1, n + 1 \). Suppose \( J_1 = I_k = [a, b] \). Since \( \bigcirc J_1 \rightarrow J_2 \Rightarrow J_2 \) is adjacent to \( J_1 \) and either

\[ x_k = a, \; x_{k+1} = f(a), \; x_{k-1} = f^2(a) \]

or

\[ x_{k+1} = b, \; x_k = f(b), \; x_{k+2} = f^2(b). \]
Consider the first case, the argument in the second being similar. If $f^3(a) < f^2(a)$ then $J_2 \rightarrow J_1$, which is forbidden. Hence $f^3(a) > f^2(a)$. Since $J_2$ is not directed to $J_k$ for $k > 3$ it follows that $J_3 = [f(a), f^3(a)]$ is adjacent to $J_1$ on the right. If $f^4(a) > f^3(a)$ then $J_3 \rightarrow J_1$, which is forbidden. Hence $f^4(a) < f^2(a)$ and, since $J_3$ is not directed to $J_k$ for $k > 4$, $J_4 = [f^4(a), f^2(a)]$ is adjacent to $J_2$ on the left. Proceeding in this way we see that the order of the $J_i$s on the real line is given by

$$
\begin{array}{cccccccc}
J_{n-1} & \cdots & J_4 & J_2 & J_1 & J_3 & \cdots & J_{n-2} \\
 f^{n-1}(a) & f^{n-3}(a) & f^4(a) & f^2(a) & a & f(a) & f^3(a) & \cdots & f^{n-4}(a) & f^{n-2}(a)
\end{array}
$$

Since the endpoints of $J_{n-1} = [x_1, x_2]$ are mapped into $a$ and $f^{n-2}(a) = x_n$ we have $J_{n-1} \rightarrow J_k$ iff $k$ is odd. We found all the arcs in the digraph. $\square$