Topological Dynamics and Universality in Chaos I. Proof of Sharkovski's Theorem

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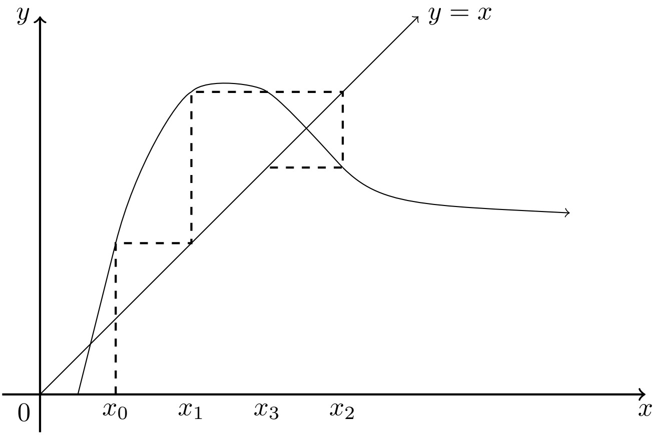
Introduction

Let $f : I \to I$ be a continuous map, and I be an interval. *Interval* is a connected subset of the real line which contains more than one point. $< a, b >$ is a closed interval with endpoints a and b.

$$f^1 = f,\ f^{n+1} = f \circ f^n,\ n \geq 1$$



Sharkovski's Theorem

$\{f^n(x) : n \geq 0\}$ be an orbit of x. $c \in I$ is a fixed point if $f(c) = c$. If $I = [a, b]$ is compact $\Rightarrow \exists$ a fixed point $c \in I$, since

$$f(a) - a \geq 0 \geq f(b) - b$$

A point $c \in I$ is said to be periodic point of f with period m if $f^m(c) = c$, $f^k(c) \neq c$ for $1 \leq k < m$. Periodic m-orbit is

$$c, f(c), f^2(c), \cdots, f^{m-1}(c)$$

Theorem 1

(Sharkovsky, 1964) Let the positive integers be totally ordered in the following way:

$$1 \succ 2 \succ 2^2 \succ 2^3 \succ ... \succ 2^2{\cdot}5 \succ 2^2{\cdot}3 \succ ... \succ 2{\cdot}5 \succ 2{\cdot}3 \succ ... \succ 7 \succ 5 \succ 3$$

If f has a cycle of period n and $m \succ n$, then f also has a periodic orbit of period m.

- A.N.Sharkovsky, Coexistence of Cycles of a Continuous Transformation of a Line into itself *Ukrain. Mat. Zhurn.*, **16**, 1(1964), 61-71.
- L.S. Block, J. Guckenheimer, M. Misiurewich, L.S. Young, Periodic Points and Topological Entropy of One-dimensional Maps, *Global Theory of Dynamical Systems*, Proc. International Conf., Northwestern Univ., Evanston, Ill.,1979, 18-34.
- L.S.Block, W.A.Coppel, Dynamics in One Dimension, Springer-Verlag, Berlin, 1992.

Proof of Sharkovski's Theorem

Lemma 2

If J is a compact subinterval such that $J \subseteq f(J)$, then f has a fixed point in J.

Proof. If $J = [a,b]$ then for some $c, d \in J$ we have $f(c) = a$, $f(d) = b$. Thus $f(c) \leq c$, $f(d) \geq d$, and by the intermediate value theorem $\exists c_* \in [a,b]\ f(c_*) = c_*$. □

Lemma 3

If J, K are compact subintervals such that $K \subseteq f(J)$, then there is a compact subinterval $L \subseteq J$ such that $f(L) = K$.

Proof. Let $K = [a,b], c = \sup\{x \in J : f(x) = a\}$. If $f(x) = b$ for some $x \in J$ with $x > c$, let d be the least and take $L = [c,d]$. Otherwise $f(x) = b$ for some $x \in J$ with $x < c$. Let c' be the greatest and let $d' \leq c$ be the least $x \in J$ with $x > c'$ for which $f(x) = a$. Then we can take $L = [c', d']$. □

Lemma 4

If $J_0, J_1, ..., J_m$ are compact subintervals such that $J_k \subseteq f(J_{k-1})$ $(1 \leq k \leq m)$, then there is a compact subinterval $L \subseteq J_0$ such that $f^m(L) = J_m$ and $f^k(L) \subseteq J_k$ $(1 \leq k < m)$.
If also $J_0 \subseteq J_m$, then there exists a point y such that $f^m(y) = y$ and $f^k(y) \in J_k$ $(0 \leq k < m)$.

Proof. Note that the first assertion holds for $m = 1$ due to Lemma 3:

$$J_1 \subseteq f(J_0) \Rightarrow \exists L \subseteq J_0, f(L) = J_1$$

Prove by induction: let $m > 1$ be fixed and assume the assertion is true for all smaller values. We have

$$J_1 \subseteq f(J_0), J_2 \subseteq f(J_1), ..., J_{m-1} \subseteq f(J_{m-2}), J_m \subseteq f(J_{m-1})$$

Induction assumption applied to last $m - 1$ relations $\Rightarrow$

$$\exists L' \subseteq J_1 : f^{m-1}(L') = J_m, f^k(L') \subseteq J_{k+1}, 1 \leq k < m - 1$$

Proof of Sharkovski's Theorem

Now we have $L' \subseteq J_1 \subseteq f(J_0) \Rightarrow \exists L \subseteq J_0$ such that $f(L) = L' \subseteq J_1 \Rightarrow$

$$f^m(L) = f^{m-1}(L') = J_m; \; f^k(L) = f^{k-1}(L') \subseteq J_k, 2 \leq k < m$$

Hence first assertion of the lemma is proved.
If also $J_0 \subseteq J_m$, from the first assertion $\Rightarrow$

$$\exists L \subseteq J_0 : \; f^m(L) = J_m \supseteq J_0 \supseteq L$$

and by Lemma 2 we have

$$\exists c \in L \subseteq J_0 : \; f^m(c) = c, \text{and } f^k(c) \subseteq J_k, 1 \leq k < m$$

Lemma 4 is proved. □

Proof of Sharkovski's Theorem

Lemma 5

Between any two points of a periodic orbit of period $n > 1$ there is a point of a periodic orbit of period less than n.

Proof. Let $a < b$ are two adjacent points of n-orbit. Consider all $m < n$ such that $f^m(b) < b$. There is at least one such m. Since there is one more point of the orbit to the left of b than to the left of a, for some m such that $1 \leq m < n$ we have $f^m(a) > a$, $f^m(b) < b$. If f^m is defined on $[a, b]$, then $\exists c \in (a, b)\ f^m(c) = c$.
Assume f^m is not defined throughout $[a, b]$. Let $J_k =< f^k(a), f^k(b) >$, $1 \leq k \leq m$. We have

$$J_k \subseteq f(J_{k-1}),\ 1 \leq k \leq m$$

We also have $J_0 \subseteq J_m$, since $f^m(a) \geq b$, $f^m(b) \leq a$. hence, the assertion of the Lemma follows from Lemma 4. □

Proof of Sharkovski's Theorem

Let $\mathbf{B} = \{x_1 < x_2 < \cdots < x_n\}$ be n-orbit of f.

Definition 1.1

If $f(x_i) = x_{s_i}, 1 \leq s_i \leq n, i = 1, 2, ..., n$, then $\mathbf{B}$ is associated with cyclic permutation

$$\begin{bmatrix} 1 & 2 & \dots & n \\ s_1 & s_2 & \dots & s_n \end{bmatrix}$$

Definition 1.2

Let $I_i = [x_i, x_{i+1}]$. Digraph of a cycle is a directed graph of transitions with vertices $I_1, I_2, \cdots, I_{n-1}$ and oriented edges $I_i \to I_s$ if $I_s \subseteq f(I_i)$.

Proof of Sharkovski's Theorem

Properties of Digraphs:

1. $\forall\ I_j\ \exists$ at least one I_k for which $I_j \to I_k$. Moreover, it is always possible to choose $k \neq j$, unless $n = 2$.
2. $\forall\ I_k\ \exists$ at least one I_j for which $I_j \to I_k$. Moreover, it is always possible to choose $j \neq k$, unless n is even and $k = \frac{n}{2}$.
   Proof: Suppose there is no $j \neq k$ such that $I_j \to I_k$. Then if $i \neq k$, $f(x_i) \leq x_k \Rightarrow f(x_{i+1}) \leq x_k$ and $f(x_i) \geq x_{k+1} \Rightarrow f(x_{i+1}) \geq x_{k+1}$. If $f(x_{k+1}) \geq x_{k+1} \Rightarrow f(x_i) \geq x_{k+1}$ for $k < i \leq n \Rightarrow$ proper subset $\{x_{k+1}, ..., x_n\}$ is mapped to itself. Hence, $f(x_{k+1}) \leq x_k$, and similarly $f(x_k) \geq x_{k+1}$, and accordingly there is a loop $I_k \to I_k$.
   $$f(x_i) \leq x_k, k < i \leq n \ \Rightarrow n - k \leq k \Rightarrow n \leq 2k,$$
   $$f(x_i) \geq x_{k+1}, 1 \leq i \leq k \Rightarrow k \leq n - k \Rightarrow n \geq 2k \Rightarrow n = 2k \quad \square$$
3. Digraph always contains a loop
   We have $f(x_1) > x_1, f(x_n) < x_n$. Let
   $$k = \min\{1 \leq j < n : f(x_j) \geq x_{j+1}, f(x_{j+1}) \leq x_j\}$$
   Then $I_k \to I_k$.

Definition 1.3

Given n-orbit, a cycle

$$J_0 \to J_1 \to \cdots \to J_{n-1} \to J_0$$

of length n in the digraph is called a **Fundamental Cycle** *(FC) if J_0 contains an endpoint c s.t. $f^k(c)$ is an endpoint of J_k for $1 \leq k < n$.*

FC always exists and unique. In the FC some vertex must occur at least twice among $J_0, ..., J_{n-1}$, since digraph has only $n-1$ vertices. On the other hand, every vertex occurs at most twice, since interval I_k has two endpoints.

Definition 1.4

Cycle in a digraph is said to be primitive if it does not consisit entirely of a cycle of smaller length described several times.

If FC contains I_k twice then it can be decomposed into two cycles of smaller length, each of which contains I_k once, and consequently is primitive.

Lemma 6

Suppose f has a periodic point of period $n > 1$. If the associated digraph contains a primitive cycle

$$J_0 \to J_1 \to \cdots \to J_{m-1} \to J_0$$

of length m, then f has a periodic point y of period m such that $f^k(y) \in J_k (0 \leq k < m)$.

Proof. $J_1 \subseteq f(J_0), J_2 \subseteq f(J_1), \cdots, J_{m-1} \subseteq f(J_{m-2}), J_0 \subseteq f(J_{m-1})$
Lemma 4 (w. $J_m = J_0$) $\Rightarrow \exists\, y \in J_0\ f^m(y) = y$, $f^k(y) \in J_k (0 \leq k < m)$
Either m is a period of y or period of y is a factor of m. If y is not an endpoint of J_0, then m is a period of y since cycle is primitive. Assume y is an endpoint of J_0. Since y is an element of n-orbit $\Rightarrow$ n is a divisor of m. We have $J_k \subseteq f(J_{k-1})$ and $f^k(y) \in J_k \Rightarrow J_k$ is defined uniquely and moreover, cycle is a multiple of the FC. Contradiction, since cycle is primitive. □

Proof of Sharkovski's Theorem

Straffin's Lemma $\Rightarrow 3 \prec m \prec 2 \prec 1$

Suppose f has a 3-orbit: $f(c) < c < f^2(c)$ with corresponding digraph

$$\circlearrowright I_1 \rightleftarrows I_2$$

$I_1 \to I_1 \Rightarrow$ there is a fixed point; $I_1 \to I_2 \to I_1 \Rightarrow$ there is a 2-orbit
$\forall$ positive integer $m > 2$ there is an m-orbit corresponding to primitive cycle of length m: $I_1 \to I_2 \to I_1 \to I_1 \to \cdots \to I_1$

Lemma 7

If f has a periodic point of period > 1, then it has a fixed point and a periodic point of period 2.

Proof. Digraph has a loop $\Rightarrow$ there is a fixed point. Let $n > 1$ be the least positive integer such that f has a periodic point of period n. If $n > 2$ decompose FC into two primitive cycles. Since at least one of these has length greater than 1, by Straffin's lemma we deduce there is a periodic point of period strictly between 1 and n. □

Proof of Sharkovski's Theorem

Stefan Orbits

Lemma 8

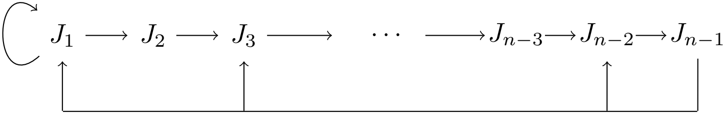
Suppose f has a periodic orbit of odd period $n > 1$, but no periodic orbit of odd period strictly between 1 and n. If c is the midpoint of the orbit of odd period n, then the points of this orbit have the order

$$f^{n-1}(c) < f^{n-3}(c) < \cdots < f^2(c) < c < f(c) < \cdots < f^{n-2}(c)$$

or the inverse order

$$f^{n-2}(c) < \cdots < f(c) < c < f^2(c) < \cdots < f^{n-3}(c) < f^{n-1}(c)$$

and associated digraph is given in the figure, where $J_1 =< c, f(c) >$ and $J_k =< f^{k-2}(c), f^k(c) >$ for $1 < k < n$.



Proof. Decompose FC into two primitive cycles, one of which has odd length. Its length must be 1! Thus FC has the form

$$J_1 \to J_1 \to J_2 \to \cdots \to J_{n-1} \to J_1$$

where $J_i \neq J_1$ for $1 < i < n$. If $J_i = J_k$, $1 < i < k < n$, then we obtain a smaller primitive cycle, and by excluding the loop at J_1 if necessary, we can arrange that its length is odd. Hence, $J_1, ..., J_{n-1}$ are all distinct and thus a permutation of $I_1, ..., I_{n-1}$. Similarly, we cannot have $J_i \to J_k$ if $k > i + 1$ or if $k = 1$ and $i \neq 1, n + 1$. Suppose $J_1 = I_k = [a, b]$. Since $\circlearrowleft J_1 \to J_2 \Rightarrow J_2$ is adjacent to J_1 and either

$$x_k = a,\ x_{k+1} = f(a),\ x_{k-1} = f^2(a)$$

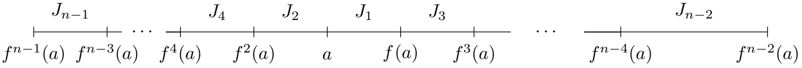
or

$$x_{k+1} = b,\ x_k = f(b),\ x_{k+2} = f^2(b).$$

Proof of Sharkovski's Theorem

Proof of Lemma 8

Consider the first case, the argument in the second being similar. If $f^3(a) < f^2(a)$ then $J_2 \to J_1$, which is forbidden. Hence $f^3(a) > f^2(a)$. Since J_2 is not directed to J_k for $k > 3$ it follows that $J_3 = [f(a), f^3(a)]$ is adjacent to J_1 on the right. If $f^4(a) > f^3(a)$ then $J_3 \to J_1$, which is forbidden. Hence $f^4(a) < f^2(a)$ and, since J_3 is not directed to J_k for $k > 4$, $J_4 = [f^4(a), f^2(a)]$ is adjacent to J_2 on the left. Proceeding in this way we see that the order of the J_is on the real line is given by



Since the endpoints of $J_{n-1} = [x_1, x_2]$ are mapped into a and $f^{n-2}(a) = x_n$ we have $J_{n-1} \to J_k$ iff k is odd. We found all the arcs in the digraph. □